Divergent Bubbles in a Small Open Economy*

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Abstract

We embed the idea of risk shifting in an asset pricing context a la Allen and Gale (2000) into a standard overlapping generations model to explain the comovement of bubbles, investment and the interest rate. To this end, we analyze a bubble in the price of a productive asset in a divergent equilibrium. The world savings fuel the bubble in the small open economy increasing the collateral value and leveraged borrowing. The bubble crowds in investment and banks charge spreads to hedge against default risks. When the bubble bursts, the interest rate collapses to the world interest rate and investment falls.

JEL Classification: E44, F41, G12
Keywords: Rational bubbles, overlapping generations model, risk shifting, small open economy, financial crisis, financial regulation

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1 Introduction

History shows that asset price bubbles play an influential role in many economic crises. We observe a typical scenario of a bubble-induced boom followed by a crash and a subsequent recession. The boom appreciates the collateral value of assets and relaxes the credit limit. Household leverage rises significantly.[1] Together with expanding credit, the bubble grows and the lending rate soars. Once the bubble crashes, the lending rate falls while the devaluation of collateral causes widespread defaults and hence financial turmoil.

Figure 1 and 2 show the growth rates of the real lending rate, the real property price index, and the new property investment during three major bubble episodes in the past three decades: Japan's asset price bubbles during 1986-91, the Asian financial crisis during 1994-97, and the US sub-prime mortgage crisis during 2003-2008.[2] Common in all three episodes is the comovement of the three variables showing similar boom-and-bust patterns with some differences in the details.[3] In particular, the lending rates of Asian countries all start to soar and reach their peaks in 1998 before they collapse. We observe similar patterns in the cases of Japan and the US.[4]

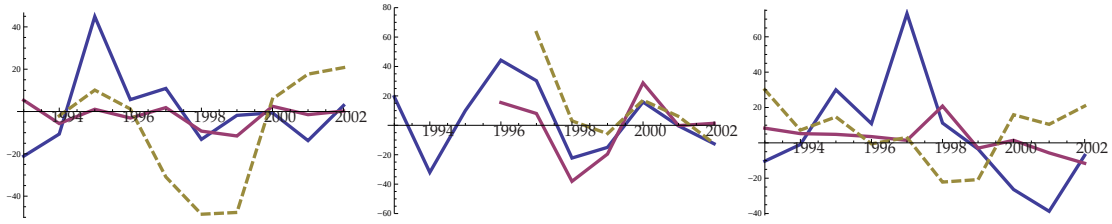
Since Tirole (1985) rational bubbles have been modeled as a store of value in the presence of shortage of financial instruments. The model by Tirole predicts that a bubble crowds out capital investment because they compete in the portfolio of investors. Empirical observations have challenged this view. Since then, fi-

[1] See Mian and Sufi (2009,2010) for U.S. example.

[2] Source: Datastream Advance, Thomson Reuters. The prime rate is used for the lending rate. For the new investment of (non)-residential buildings, "GFCF: PRIVATE BUILDING CONSTRUCTION" is used for Thailand, "HOUSING APPROVED" is used for Malaysia, "GFCF: NON-RESIDENTIAL BUILDINGS" is used for South Korea, "PRIVATE NON-RESIDENTIAL INVESTMENT" is used for Japan, and "GFCF: PRIVATE NON-RESIDENTIAL" is used for the US. For the price of (non)-residential buildings, "HOUSE PRICE INDEX" is used for Thailand, Malaysia, South Korea and Japan, and "PPI: INPUTS TO CONSTRUCTION, NON-RESIDENTIAL" is used for the US. "CORE CPI" is used for every country except Malaysia where overall CPI is used.

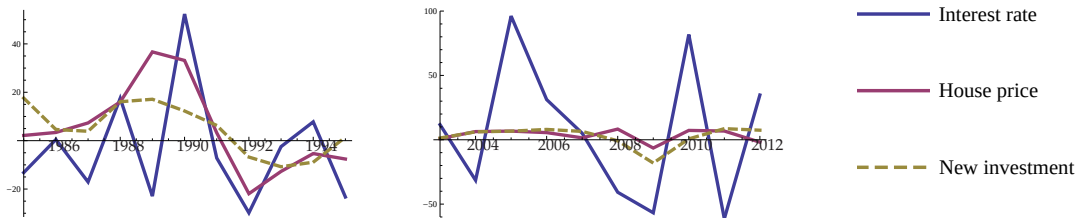
[3] Even though the real property price in Thailand did not grow in level preceding the crisis, it is apparent from the figure that it declined sharply together with the real interest rate during the crisis.

[4] In the US the real lending rate dropped but rebounded earlier than the real house price making it appear that the burst of the bubble triggered an increase in the real lending rate.



(a) Thailand (b) Malaysia (c) South Korea

Figure 1: The Asian financial crisis



(a) Japan (b) US

Figure 2: Japan's asset price bubble and the US sub-prime mortgage crisis

nancial frictions have been introduced into models of rational bubbles to explain investment booms and bubbles. Major contributions include Caballero and Krishnamurthy (2006), Caballero, Farhi and Hammour (2006), Kocherlakota (2009), Farhi and Tirole (2012), Martin and Ventura (2011, 2012) and Ventura (2012).

These contributions, however, do not show a comovement of the three variables: bubbles, investment and the interest rate. For example, in Caballero and Krishnamurthy (2006), the domestic lending rate stays constant at a low level in a bubble boom and jumps to a high level when a bubble bursts since the burst of the bubble reduces the wealth and hence increases the net domestic demand for loans. In Caballero, Farhi and Hammour (2006), bubbles can emerge along a speculative equilibrium and co-move with investment but not with the interest rate. Moreover, as in Tirole (1985), a bubble crowds out investment and leads to a lower capital stock relative to the bubbleless equilibrium in their model. In Farhi and Tirole (2012) a bubbly steady state is associated with a higher investment than in a bubble-less steady state. On the stable manifold, however, the size of bubbles either shrinks with a rising investment or expands with a declining investment.

Models of rational bubbles typically consider a bubble of an unproductive assets that converges to a steady state value. When people can purchase an unproduc-

tive bubble and invest in productive technology, they must be indifferent between the two. Therefore, the no-arbitrage condition $B_{t+1}/B_t = 1 + r_{t+1}$, where B_t is the bubble and r_t is the interest rate at time t, holds. As a result, when a bubble grows in transition to a steady state, it must increase with a decreasing rate approaching a constant value. The decreasing rate implies that the interest rate must fall and converge to zero. In other words, capital stock must increase to drive down the marginal product of capital and hence the interest rate—a negative correlation between the bubble and the interest rate.

In the context of economic growth Olivier (2001) and Hirano and Yanagawa (2013) show that bubbles raise the interest rate but can crowd in investment. This happens in Olivier (2001) when bubbles on equity create incentive for R&D activities that dominate the rising cost for borrowing, and in Hirano and Yanagawa (2013) when bubbles increase the net worth of entrepreneurs and thus future investment.

We take a totally different modeling approach. In a highly influential paper Allen and Gale (2000) showed that bubbles can be caused by agency relationship in the banking sector. Investors borrow from banks to invest in risky assets but can avoid losses in low payoff states by defaulting on the loan. This risk shifting leads investors to bid up the asset prices. Debt contracts are formulated in which banks charge spreads to hedge against default risks when a bubble arises. We embed the idea of risk shifting a la Allen and Gale (2000) into a standard overlapping generations model to explain the comovement of bubbles, investment and the interest rate.

We analyze a bubble in the price of an investment good in a divergent equilibrium. People in a small open economy have access to unlimited resources in the world financial market but can only pledge investment goods as collateral when taking loans.[5] A bubble pushes up the credit limit by increasing the collateral value. The bubble then grows together with the credit volume and crowds in in-

[5]The interplay between the credit constraint and the investment in the collateralized asset has been discussed extensively in the literature. In their seminar paper Kiyotaki and Moore (1997) show that the feedback loop magnifies effects of any perturbation in the economy. Challe and Ragot (2011) discuss the problem of risk-shifting when agents cannot write a state-contingent debt contract. They show that the creditor has to bear the risk of the investment when the debtor's liability is limited by the pledgeable income. This protection against the bad payoff may lead to over-investment and asset price bubbles.

vestment as production of new investment goods increases with the rising price. Loans are increasingly backed by the bubble as the gap between the "bubbly" and the fundamental price of the investment good increases. This means that the loss of collateral in the event that the bubble bursts is increasing in the bubble. Hence, the lending rate (and thus the interest rate spread) increases on a divergent path as the bubble and investment rise.[6] Once the bubble bursts, the collateral value, the lending rate and investment drop to their fundamental levels.

Our model advises caution in adopting a regulatory policy to eliminate bubbles. In the presence of the collateral constraint, the positive feedback between a bubble and the supply of a credit is driven by a financial accelerator—a balance-sheet effect—as a bubble raises the collateral value and thus the supply of a credit.[7] To eliminate bubbles, this linkage must be cut. An arbitrary regulation, however, may be too restrictive and hurt fundamentals of the economy. Only when loans are restricted to the fundamental value of the collateral, bubbles are ruled out in our model without suppressing fundamentals in a similar manner as the natural credit limit dampens the financial accelerator in Aiyagari (1994).

The paper is organized as follows. Section 2 outlines the economy. Section 3 analyzes the equilibrium of the economy and defines the fundamental price. The sunspot equilibrium is constructed in Section 4. Section 5 describes the boom-bust scenario of bubbles. Section 6 considers regulatory policies. Section 7 generalizes the small open economy analysis. Section 8 concludes.

2 Setup

We consider a small open economy populated by overlapping generations who live for two periods. The economy faces a constant world interest rate $r^* \in \mathbb{R}_+$. All markets are competitive. There are two types of goods: a consumption good and an investment good. The consumption good is the numeraire and the price

[6]The interest rate spread for South Korea and Thailand increased before 1997 and the risk premium on lending increased for Malaysia before 1997, for Japan before 1991 and for US before 2007. Source: The World Development Indicators.

[7]In the context of the Asian currency crisis Krugman (1999) points out the importance of the balance-sheet effect in which there is a two-way dependency between the ability of a firm to borrow and the actual borrowing.

is set to 1 while the relative price of the investment good in period t is denoted as p_t. Each generation consists of two types of agents: type 1 (*contractors*) and type 2 (*producers*). Each type has unit mass and is endowed with consumption goods $w > 0$ when young and consumes only when old.[8] There is no population growth.

Young contractors produce investment goods $f(y_t)$ out of consumption goods y_t with a neoclassical technology: $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, strictly concave, twice continuously differentiable on $\mathbb{R}_{++}$, $f(0) = 0$, $\lim_{k\uparrow\infty} f'(y) = 0$, and $\lim_{y\downarrow 0} f'(y) = \infty$. Consumption goods fully depreciate after production. Old contractors in period t sell investment goods x_t at price p_t to young producers who produce consumption goods available in the following period. Young producers make one unit of consumption goods out of one unit of investment goods—for simplification. The investment good depreciates at rate $\theta \in (0, 1)$ after production.[9]

Everyone can borrow and lend in the financial market. The borrowing of contractors $b_{1,t}$ and of producers $b_{2,t}$ is given by

$$b_{1,t} = y_t - w \tag{1}$$

$$b_{2,t} = p_t x_t - w. \tag{2}$$

Everyone can only pledge the investment goods when borrowing from the market.[10] Therefore, the credit is limited up to the future value of the investment goods they hold:

$$(1 + r_{1,t+1})(y_t - w) \leq p_{t+1} f(y_t) \tag{3}$$

$$(1 + r_{2,t+1})(p_t x_t - w) \leq p_{t+1}(1 - \theta) x_t \tag{4}$$

where $r_{1,t+1}$ and $r_{2,t+1}$ are the lending rates for contractors and producers.

[8] We rule out the case $w = 0$ since in this case the only steady state is zero and no production and trade takes place.

[9] The assumption that $\theta \notin \{0, 1\}$ is crucial for existence of a stable manifold that defines the fundamental price in Proposition 1. If $\theta = 0$, then the stock of investment goods would not decrease. If $\theta = 1$, then the equilibrium dynamics collapses to the steady state: a full depreciation of investment goods makes the stock of investment goods a jump variable.

[10] Borrowers may intentionally default by misreporting their incomes. In this case, banks may only verify investment goods, which are durable and immovable (e.g. factory buildings) as collateral. This assumption will play a key role in Section 7 when we introduce growth (technological progress) in the consumption goods sector.

3 Equilibrium

There exist both deterministic and stochastic equilibria. This section focuses on the deterministic equilibria.

There are a large number of competitive risk-neutral banks that have access to savings in the international financial market at the constant interest rate r^*. Competition forces banks to lend at the interest rate equal to the cost of funds:

$$r_{1,t+1} = r_{2,t+1} = r^*. \tag{5}$$

The credit constraint never binds for contractors who can pledge their entire future revenue. Young contractors choose the input of consumption goods by solving

$$\max_y \{p_{t+1} f(y) - (1 + r^*)(y - w)\}.$$

The first order condition yields

$$p_{t+1} f'(y_t) = 1 + r^*. \tag{6}$$

The condition shows that contractors may produce an arbitrary large amount of the investment good even in the presence of diminishing marginal product when its price becomes arbitrary large. Solving for y_t we obtain the demand of consumption goods for the production of investment goods at time t:

$$y_t = f'^{-1}\left(\frac{1 + r^*}{p_{t+1}}\right). \tag{7}$$

Given x_t and y_t the stock of investment goods evolves according to

$$x_{t+1} = (1 - \theta)x_t + f(y_t) \tag{8}$$

where the first term on the right hand side is the depreciated stock hold by producers and the second term is the new stock produced by contractors.

Young producers choose the input of investment goods by solving

$$\max_x \{p_{t+1}(1 - \theta)x + x - (1 + r^*)(p_t x - w)\}.$$

The first order condition yields

$$p_{t+1}(1 - \theta) + 1 = (1 + r^*)p_t. \tag{9}$$

The equation shows that producers would determine the investment goods input for consumption goods production at the point where the rate of return from the production and selling the depreciated input after production is equal to the world interest rate. However, this may not always hold because they cannot pledge the consumptions goods output. Therefore, we need to take into account the credit constraint:

$$(1+r^*)(p_t x_t - w) \leq p_{t+1}(1-\theta)x_t. \tag{10}$$

Combining (9) and (10) the price of investment goods evolves according to

$$p_{t+1} = \frac{(1+r^*)p_t - \min\{1, (1+r^*)w/x_t\}}{1-\theta}. \tag{11}$$

When is the credit constraint binding? The future price of investment goods is bounded above by the current price and the world interest rate through the no-arbitrage condition (9). On the other hand, producers' demand for loans increases with the stock of investment goods. As a result, the collateral value does not catch up with the demand for loans and the credit constraint binds when the stock of investment goods is sufficiently high:

$$x_t \geq \hat{x} := (1+r^*)w.$$

We now define a bubble in the price of the investment good. By iterating (11) the price of the investment good can be decomposed into its fundamental value and a bubble element:

$$p_t = \sum_{i=0}^{\infty} \left(\frac{1-\theta}{1+r^*}\right)^i \frac{\min\{1, (1+r^*)w/x_{t+i}\}}{1+r^*} + \lim_{k\to\infty} \left(\frac{1-\theta}{1+r^*}\right)^k p_{t+k}.$$

The first term on the right-hand-side represents the fundamental value and the second term the bubble element. Note that the first term may be lower than the sum of discounted stream of the marginal products of investment goods because of the credit constraint.

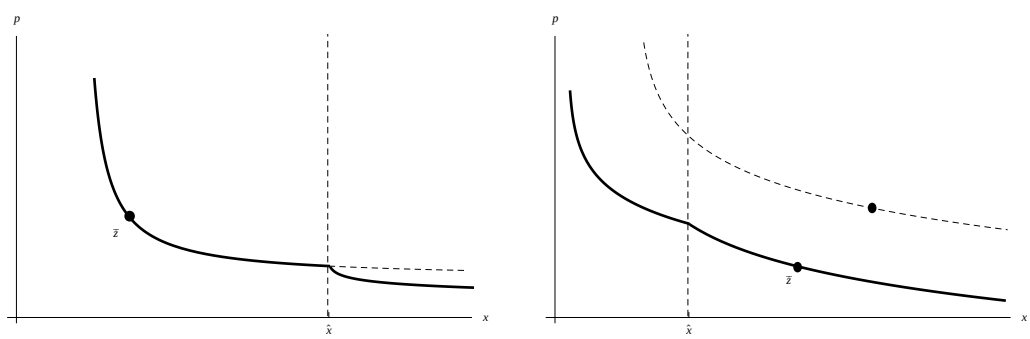
We are now ready to analyze the dynamics. Let $z_t := (p_t, x_t)$. Equations (7), (8) and (11) define a dynamical system $z_{t+1} = \phi(z_t)$.

Lemma 3.1. *There exists a unique positive steady state $\bar{z} := (\bar{p}, \bar{x})$.*

Definition 3.1. A function $\rho : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is called a fundamental price function if for any $x_t \in \mathbb{R}_{++}$, $(\rho(x_{t+1}), x_{t+1}) = \phi(\rho(x_t), x_t)$ and $\lim_{k\to\infty} \left(\frac{1-\theta}{1+r^*}\right)^k p_{t+k} = 0$.

The fundamental price function defines the bubble-free equilibrium price. Once the investment good is priced at $\rho(x_t)$, the price in following periods will no longer contain a bubble element. We will refer to $\rho(x_t)$ as the fundamental price of the investment good x_t. Any other price will be called a bubble. Correspondingly we will call an equilibrium fundamental when $p_t = \rho(x_t)$ for all t and bubbly when $p_t \neq \rho(x_t)$ for all t. Let $\phi^{\{n\}}$ be the n-time iteration of ϕ. The next theorem gives the property of the fundamental price function.

Theorem 3.1. *The system ϕ is a saddle. Its global stable manifold defines a unique fundamental price function $\rho(x)$, which is continuous and strictly decreasing and satisfies $\lim_{n\to\infty} \phi^{\{n\}}(\rho(x), x) = \bar{z}$.*



(a) $\bar{x} < \hat{x}$ (b) $\bar{x} > \hat{x}$

Figure 3: Fundamental price function

The solid line in Figure 3 depicts the fundamental price function. The dashed line depicts the fundamental price function in the absence of the credit constraint. The figures show that the credit constraint decreases the fundamental price: it leads to a lower demand and a lower price, and suppresses the fundamentals of the economy.

Since ϕ is a saddle, any path under the fundamental price eventually violates the non-negativity of the price and is ruled out. The following corollary shows that any path of the price above the fundamental price level is a bubble.

Theorem 3.2. *Any price higher than the fundamental price is a bubble. In other words, if $p_t > \rho(x_t)$ for any (p_t, x_t), then $0 < \lim_{k\to\infty} \left(\frac{1-\theta}{1+r^*}\right)^k p_{t+k} < \infty$.*

The theorem implies that bubbles emerge and crowd in investment in a divergent equilibrium as $(p_t, x_t) \to \infty$.

4 Sunspot equilibrium

So far we have shown that the equilibrium can be either a fundamental equilibrium or a bubbly equilibrium. The fundamental equilibrium is obtained when the initial condition lies on the stable manifold—$p_0 = \rho(x_0)$. The bubbly equilibrium is obtained when the initial condition lies anywhere above the stable manifold—$p_0 > \rho(x_0)$. We now construct the sunspot equilibrium where bubbles burst with a probability $q \in (0, 1)$ and continue to grow with a probability $1 - q$ each period; bubbles never emerge in the future once they burst. This once-and-for-all crash ensures that the fundamental equilibrium can be used as the absorbing state when bubbles burst. The bubble with a possibility of bursting is called a stochastic bubble in the literature.[11]

With a stochastic bubble, all micro-level optimal conditions need to be modified. For banks, the debt contract has to be rewritten considering the potential default risk. We assume that borrowers repay their debt if the realized value of their pledgeable income exceeds their debt obligation; otherwise they default and lose their collateral. In other words, banks ensure through the credit constraints that borrowers have an incentive to repay as long as the bubble continues, while banks accept that borrowers may default when the bubble bursts. Let $C_1 := \rho(x_{t+1})f(y_t)$ and $C_2 := (1-\theta)\rho(x_{t+1})x_t$ denote the fundamental value of the collateral held by contractors and producers. The optimal debt contract is then given by

$$(1-q)(1+r_{j,t+1})b_{j,t} + q\min\{(1+r_{j,t+1})b_{j,t}, C_j\} = (1+r^*)b_{j,t} \quad \forall j = 1, 2 \quad (12)$$

where (3) and (4) hold.

If the fundamental value of the collateral covers the debt obligation, the debt is repaid in both states of the world, and the lending rate is equal to the world interest rate: if $(1+r_{j,t+1})b_{j,t} \leq C_j$, then $r_{j,t+1} = r^*$. If the debt obligation exceeds the fundamental value of the collateral, the debtors default in the pessimistic

[11] The notion of stochastic bubbles in general equilibrium was first introduced in Weil (1987).

state, and therefore banks charge a lending rate higher than the world interest rate: if $(1 + r_{j,t+1})b_{j,t} > C_j$, then $r_{1,t+1} > r^*$. Banks raise the lending rate until the debt obligation equals the market value of the collateral. If banks charge a higher interest rate, the debtor would default even in the optimistic state.

Lemma 4.1. *Banks charge a lending rate higher than the world interest rate if and only if the loan exceeds the fundamental value of the collateral discounted by the world interest rate:* $b_{j,t} > C_j/(1 + r^*)$.

Replacing p_{t+1} with $\mathbb{E}_t(p_{t+1}) := (1 - q)p_{t+1} + q\rho(x_t)$ in (8) and (11) we obtain the evolution of the price and stock of investment goods. Let $z_{t+1} = \varphi(z_t)$ denote the new dynamical system.

Definition 4.1. Given $x_0 \in \mathbb{R}_{++}$, $p_0 \geq \rho(x_0)$ and a crash time period T, a sunspot equilibrium is a sequence $\{p_t, x_t\}_{t=0}^{T-1}$ that satisfies φ, followed by a sequence $\{p_t, x_t\}_{t=T}^{\infty}$ that satisfies ϕ and $(p_T, x_T) = (\rho(x_T), x_T)$.

Proposition 4.1. *Under the system φ, given $x_0 \in R_{++}$ and $p_0 \geq \rho(x_0)$, there exists a threshold function $\hat{\rho}(x)$ satisfying the following properties:*

1. $\hat{\rho}(x) \geq \rho(x)$ *for all* $x \in \mathbb{R}_{++}$.
2. $\hat{\rho}(x)$ *is continuous, strictly decreasing over* $(0, \bar{x}]$ *and strictly increasing over* $(\bar{x}, \infty)$.
3. *For any* $t \leq T - 1$, $p_{t+1} > p_t$ *and* $x_{t+1} > x_t$ *if and only if* $p_t > \hat{\rho}(x_t)$.
4. *Given* $p_0 > \rho(x_0)$ *and a sufficiently large* T, *there exists* $\hat{t} < T - 1$ *with* $p_{\hat{t}} > \hat{\rho}(x_{\hat{t}})$, $p_{t+1} > p_t$, *and* $x_{t+1} > x_t$ *for all* $\hat{t} \leq t \leq T - 1$.

Proposition 4.1 implies that if the initial price is above the fundamental price level, the economy is in a divergent equilibrium and the price and stock of investment goods eventually increase monotonically. Stochastic bubbles are expected to crash onto the fundamental price and the small open economy sustains bubbles by absorbing international savings. Young agents are willing to purchase the bubbles in the hope that the crash will not occur—-bubbles reflect speculative investment motives.

5 Bubbles, investment and interest rates during boom and bust

We are now ready to describe the lending rate dynamics and see how it moves together with bubbles and investment during boom and bust. It turns out that we can define the prices at which the debt obligation is equal to the fundamental value of collateral $(1 + r^*)b_j = C_j$ for $j = 1, 2$. We denote $p_t = l_1(x_t)$ and $p_t = l_2(x_t)$ the price for contractors and producers.

Assumption 1. $\frac{kf'(k)}{f(k)} < 1$.

The assumption ensures that the term $\frac{k}{f(k)}$ is strictly increasing which is crucial for l_1 to be decreasing.

Lemma 5.1.

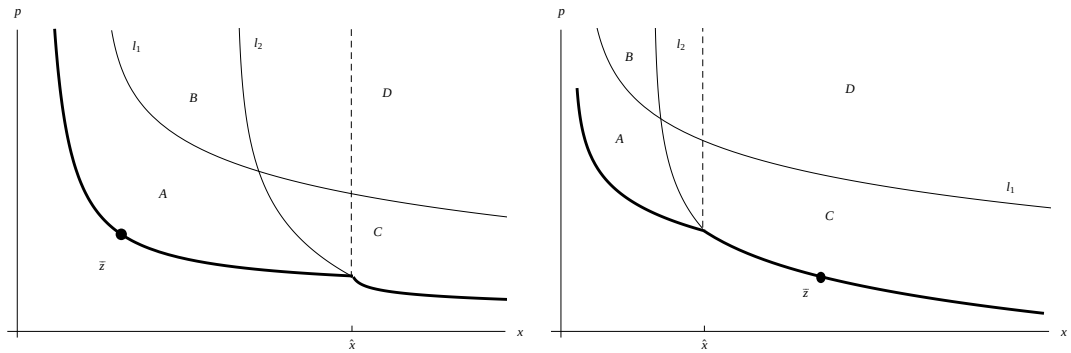
1. *$l_1(x) \geq \rho(x)$ and $l_1(x)$ is strictly decreasing.*
2. *$l_2(x) \geq \rho(x)$ for $x \in (0, \hat{x}]$, $l_2(x) = \rho(x)$ for $x \in (\hat{x}, \infty)$, and $l_2(x)$ is strictly decreasing.*

Lemma 4.1 together with Lemma 5.1 leads to the next theorem.

Theorem 5.1. *As bubbles continue to grow and crowd in investment, banks start to take more default risks and hence increase the lending rate beyond the world interest rate. As $(p_t, x_t) \to \infty$, $1 + r_{j,t} \to \frac{1+r^*}{1-q}$. When bubbles burst, the lending rate falls down to the world interest rate level and investment falls.*

Figure 4 and Table 1 illustrate the findings.

The claim of the theorem follows from Proposition 4.1 and Lemmas 4.1 and 5.1. When bubbles continue to rise, the investment goods production becomes more profitable for contractors while producers need more funds to purchase investment goods; everyone increases the demand for loans and the debt obligation exceeds the fundamental value of collateral. To hedge against default risks banks raise the lending rate. The sunspot equilibrium moves out of region A, B and C in the figure when (x_t, p_t) becomes arbitrary large. When bubbles collapse to the



(a) $\bar{x} < \hat{x}$ (b) $\bar{x} > \hat{x}$

Figure 4: Lending rates in each region

Region	$r_{1,t+1}$	$r_{2,t+1}$
A	$= r^*$	$= r^*$
B	$> r^*$	$= r^*$
C	$= r^*$	$> r^*$
D	$> r^*$	$> r^*$

Table 1: Lending rates in each region

fundamental price, a boom turns into a bust. Everyone defaults on their debt and banks receive the devalued collateral at the fundamental price.[12] If the stock of investment goods is above the steady state level at the time of the collapse, investment falls and converges to the steady state. Finally, $1 + r_{j,t} \to \frac{1+r^*}{1-q}$ follows from (12) because $b_j \to \infty$ if $(p_t, x_t) \to \infty$.

Now, what are the welfare effects of bubbles? Since the trajectory of a bubble depends on initial conditions, it is hard to compare the welfare of the fundamental equilibrium and the bubbly equilibrium. Nevertheless, we can examine whether each generation born in period t—given x_t—is better off holding bubbles.

[12]Our model's prediction about dynamics in the financial sector is consistent with what happened during the Asian financial crisis in 1990s: many financial institutions went bankrupt due to a widespread default and a sharp devaluation of their collateral. For example, 56 insolvent finance companies and 1 commercial bank were closed in Thailand during the crisis. Furthermore, the short-term commercial bank credit in Indonesia, Korea, Malaysia, the Philippines, and Thailand in 1997 decreased by \$77.1 billion from the 1996-level (a drop \$15.8 billion for non-bank private credit). Source: The year-1999 report of Institute of International Finance.

First, the consumption of producers is independent of the price of investment goods and therefore the same in bubbly, sunspot, or fundamental equilibrium. Producers born in period t consume in the following period

$$c_{2,t+1} = 1 - \min\left\{1, \frac{(1+r^*)w}{x_t}\right\} + (1+r^*)w.$$

Since $c_{2,t+1}$ is increasing in x_t, the consumption of producers in the fundamental equilibrium increases if and only if $x_t < \bar{x}$ while in the bubbly equilibrium it always increases as $x_t \to \infty$: producers benefit during a bubble boom since bubbles crowd in investment.

Second, the consumption of contractors depends on the price of investment goods. Contractors born in period t consume in the following period

$$c_{1,t+1} = p_{t+1} f \circ f'^{-1}\left(\frac{1+r^*}{p_{t+1}}\right) - (1+r^*)\left[f'^{-1}\left(\frac{1+r^*}{p_{t+1}}\right) - w\right]$$

where $p_{t+1} = \rho(x_{t+1})$ in a fundamental equilibrium. The consumption of contractors increases with the price as

$$\frac{dc_{1,t+1}}{dp_{t+1}} = f \circ f'^{-1}\left(\frac{1+r^*}{p_{t+1}}\right) > 0.$$

In a bubbly equilibrium, $p_{t+1} > \rho(x_{t+1})$, and therefore, contractors consume more than in a fundamental equilibrium. The result holds even when bubbles eventually burst in a sunspot equilibrium because $\mathbb{E}_t(p_{t+1}) > \rho(x_{t+1})$ holds as the economy diverges from the fundamental equilibrium. In a sunspot equilibrium, contractors consume more than the expected value when a bubble continues to grow while they consume lower than the expected value when the bubble bursts.

The only negative feature of a bubble in our model is that it bursts. To investigate whether we should prevent a bubble to emerge because of the losses when the bubble bursts is beyond the scope of this paper and left for future research. The next section only makes the first step by examining how regulatory policies affect bubbles and the fundamental price.

6 Regulatory policies

Both contractors and producers fully borrow against their pledgeable income—the future value of the investment goods they hold. As the bubble grows, the

price of investment goods rises and the credit constraint is relaxed. This positive feedback loop between the bubble and the supply of credit inflates the bubble further. Setting a credit ceiling would cut this feedback loop, stop the expanding supply of credit, and rule out bubbles. The type of policy may range from limiting borrowing to a fraction of the future value of investment goods to no borrowing at all, which could be devastating since it hurts the fundamentals of the economy. To make our point we consider two extreme regulatory policies that can eliminate bubbles.

Proposition 6.1. *If the value of credit is limited to the fundamental value of the collateral, no bubbles arise and the fundamental price is not affected.*

Theorem 3.1 and Proposition 4.1 show that bubbles eventually leads to an increase in the stock and price of investment goods. In the presence of the fundamental value constraint, people realize that bubbles will not be affordable in the future; rational expectation equilibria cannot contain bubbles. The fundamental credit constraint in Proposition 6.1, however, is hard to implement in practice since it requires the knowledge of the fundamental price function. We would like to have a regulatory policy which is implementable and able to rule out bubbles with a tolerable damage on the fundamentals.

The next proposition considers a common marked-to-market credit constraint: banks use the current price as a proxy because the future price of the collateral is not fully predictable.

Proposition 6.2. *If the value of credit is limited to the marked-to-market-value, then no bubbles arise but the fundamental price may be suppressed.*

The marked-to-market-value constraint sets a constant limit to the current investment good purchase. Hence, bubbles will be ruled out for the same reason as with the fundamental value constraint, but at the cost of suppressing parts of the fundamental price path. The marked-to-market-value constraint, however, does not suppress the fundamental steady state.[13] On the other hand, an arbi-

[13]Note that Proposition 6.1 and Proposition 6.2 do not state that the fundamental price $\rho(x)$ is the only equilibrium path. These policies may create new convergent equilibrium paths with new steady states. These paths would contain no bubbles and the new steady states would lie above $\rho(x)$. In other words, the policy could raise the long run fundamental price level of investment goods.

trary margin constraint would suppresses not only the fundamental price path but also the long-run fundamental price.

7 Growing bubbles and the world economy

So far we have taken the standard notion of a small open economy that has an infinitesimal size compared to the world: whatever happens in the small open economy it does not affect the world economy, in particular the world interest rate. The assumption, however, may be questionable in our setup where the expenditure on bubbles grows over time in a bubbly equilibrium. It is a legitimate question to ask if bubbles can eventually absorb the entire world economy. Therefore, this section generalizes the previous model to analyze conditions under which the small open economy remains infinitesimally small compared to the world. The analysis not only gives us sufficient conditions for existence of bubbles in both bubbly and sunspot equilibrium, but also reveals a hidden role of the borrowing constraint in generating bubbles.

Let us start by showing why bubbles may not be feasible in the previous setup. The relative size of the small open economy at time t is given by

$$s_t = \prod_{i=1}^{t} \frac{p_i x_i}{p_{i-1} x_{i-1}} \frac{1}{(1+g)^t} \cdot s_0 \tag{13}$$

where g is the growth rate of the world economy. We can see that if the relative size is infinitesimally small initially, i.e., $s_0 = \frac{1}{\infty}$, it remains infinitesimally small as long as the term $\prod_{i=1}^{t} \frac{p_i x_i}{p_{i-1} x_{i-1}} \frac{1}{(1+g)^t}$ remains finite as $t \to \infty$. Hence, it is sufficient to check if $\mathbb{E}_t(\lim_{t\to\infty} \frac{p_{t+1} x_{t+1}}{p_t x_t}) < 1 + g$ in order to guarantee that the small open economy remains infinitesimally small.[14]

Lemma 7.1. *Suppose that $f(y) = y^\alpha$ where $\alpha \in (0, 1)$.*

1. *In the bubbly equilibrium, $\lim_{t\to\infty} \frac{p_{t+1} x_{t+1}}{p_t x_t} = 1 + \hat{g}$ where $\hat{g}$ is a finite threshold and $\hat{g} > r^*$.*

[14]This is similar to the existence condition of bubbles in Santos and Woodford (1997) that the share of the aggregate endowment of any one agent has to tend to zero asymptotically.

2. *In the sunspot equilibrium*, $\mathbb{E}_t[\lim_{t\to\infty} \frac{p_{t+1}x_{t+1}}{p_t x_t}] = 1$.[15]

If the world economy does not grow as in the previous sections, Lemma 7.1 implies that the small open economy may outgrow the world economy in the long run in both bubbly and sunspot equilibrium. Hence, the world economy needs to grow to support non-stationary bubbles.

We could simply assume that the world economy grows faster than the small open economy. If the small open economy does not grow while the world economy grows at rate $g > \hat{g}$, the world can support bubbles in both bubbly and sunspot equilibrium. If $0 < g < \hat{g}$, the world can only support stochastic bubbles.[16] The assumption of the differential growth rates may be too strong. Hence, we now consider the case when the world economy and the small open economy grow at the same rate but only in the consumption goods sector: the output of consumption goods is given by $a_t x_t$ and $a_{t+1} = (1+g)a_t$.[17]

The deterministic dynamical system becomes

$$\begin{aligned} x_{t+1} &= (1-\theta)x_t + f \circ f'^{-1}\left(\frac{1+r^*}{p_{t+1}}\right) \\ p_{t+1} &= \frac{(1+r^*)p_t - \min\{a_t, (1+r^*)w/x_t\}}{1-\theta} \\ a_{t+1} &= (1+g)a_t. \end{aligned} \tag{14}$$

We immediately observe that the credit constraint is binding if $x_t > (1+r^*)w/a_t$. Now if we set the initial value such that $x_0 > (1+r^*)w/a_0$, the credit constraint is always binding, and the dynamics becomes independent of a_t. The next proposition follows from Lemma 7.1.

[15] This condition is sufficient for a stochastic bubble to emerge since the expenditure on bubbles at any finite time is expected to be finite and hence it remains "small" relative to the world.

[16] If the world is described by a standard Ramsey-Cass-Koopmans model with a discount factor $\beta \in (0,1)$ and a utility function $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, the steady state world interest rate would be $1 + r^* = \frac{(1+g)^\gamma}{\beta}$. If $\gamma \geq 1$, the last equation implies $g < r^*$, which is inconsistent with the result in Lemma 7.1 2. Hence, the bubbly equilibrium in a small open economy may not be sustainable. This problem can be averted by assuming some sorts of financial frictions in the world economy common in the literature.

[17] One natural modification would be to allow technologies in both consumption and investment goods sectors as well as the endowment w to grow at the same rate as the growth rate of the world economy. However, this does not change the model; the variables would only be re-normalized.

Proposition 7.1. *Suppose that $f(y) = y^{\alpha}$ where $\alpha \in (0,1)$ and $a_{t+1} = (1+g)a_t$. Given $\bar{x} > \hat{x}$ and $x_0 > \hat{x}$, the world economy can support bubbles in the bubbly equilibrium if $g > \hat{g}$ and in the sunspot equilibrium if $g > 0$.*

Now we reveal the role of the credit constraint in generating bubbles in a growing economy. Consider (14) with a sufficiently high technology growth and no credit constraint: $g > (1+r^*)/(1-\theta)$ and $a_t < (1+r^*)w/x_t$. In this case all price dynamics converge to zero and bubbles do not emerge. The imbalance between the consumption goods sector and the investment goods sector makes investment in the consumption good sector more attractive than in the investment goods sector, and thus increases the demand for investment goods. The supply of investment goods, however, cannot catch up with the demand due to its lack of technological progress. For the investment goods market to clear, the current price relative to the future price must rise, and the price converges to the trivial steady state.[18] Without the credit constraint the excessive demand for investment goods rules out bubbles.[19] The credit constraint attenuates the demand for investment goods and generates bubbles.

8 Conclusion

We embed the idea of risk shifting in an asset pricing context a la Allen and Gale (2000) into a standard overlapping generations model to explain the comovement of bubbles, investment and the interest rate. The crowding-in effects of bubbles have been extensively analyzed in the recent contributions on bubbles in an economy with financial frictions. The literature commonly analyzes bubbles in the price of an unproductive asset in a convergent equilibrium to a steady state, as a result, obtaining a negative correlation between bubbles and the interest rate.

We, on the other hand, analyze a bubble in the price of a productive asset in a divergent equilibrium. As the bubble grows, the price of the investment goods and thus new investment increase. The growing gap between the "bubbly" price and the fundamental price on a divergent path increases the loss of collateral in

[18]Such dynamics contains no bubble since $\lim_{k\to\infty}\left(\frac{1-\theta}{1+r^*}\right)^k p_{t+k} = 0$.

[19]This result is specific to the setup in this section when there is growth in the consumption goods sector.

the event that the bubble bursts. Hence, the lending rate increases on a divergent path as the bubble and investment rise: a comovement of the three variables.

When a bubbles grows, banks take higher default risks and raise the lending rate. When a bubble bursts, debtors default and the lending rate drops to the world interest rate. Banks bear losses and may go bankrupt because of the devaluated collateral. In fact, the only negative feature of a bubble in our model is that it bursts. On the other hand, eliminating bubbles through regulating provision of credit is likely to hurt the fundamentals of the economy. Many important questions remain unanswered. How do the losses induced by bursting of a bubble weigh against the benefits of a bubble—for example an increase in consumption? Should we bail out banks? What is the optimal way to do it? What role does the central bank play? We hope to answer these questions in future research.

9 Appendix

Proof of Lemma 3.1. Let $X(x) := x - \frac{1}{\theta} \cdot f \circ f'^{-1}\left(\frac{(1+r^*)(r^*+\theta)}{\min\{1,(1+r^*)w/x\}}\right)$. By definition, the steady state $(\bar{x}, \bar{p})$ must satisfy $X(\bar{x}) = 0$ and $\bar{p} = \frac{\min\{1,(1+r^*)w/\bar{x}\}}{r^*+\theta}$. We observe that $X(x)$ is continuous, strictly increasing, $\lim_{x\to 0} X(x) < 0$ and $\lim_{x\to\infty} X(x) = \infty$. Hence, we obtain a unique $\bar{x}$. □

Proof of Theorem 3.1. The system is formally defined as $\phi : \Theta \to \mathbb{R}^2_{++}$ where $\Theta \in \mathbb{R}^2_{++}$ is a neighborhood of $\bar{z}$. Let $h(x) := \min\{1, (1+r^*)w/x\}$. The characteristic equation is given by

$$\lambda^2 - \left[\frac{1+r^*}{1-\theta} + (1-\theta) + \frac{(1+r^*)\, f'(\bar{y})h'(\bar{x})}{(1-\theta)\bar{p}^2 f''(\bar{y})}\right]\lambda + 1 + r^* = 0$$

where $\bar{y} := f'^{-1}\left(\frac{1+r^*}{\bar{p}}\right)$.

Let

$$\varpi := \frac{1}{2}\left[\sqrt{\Omega^2 - 4(1+r^*)} - \Lambda\right]$$

where

$$\Omega := \frac{1+r^*}{1-\theta} + (1-\theta) + \frac{(1+r^*)f'(\bar{y})h'(\bar{x})}{(1-\theta)\bar{p}^2 f''(\bar{y})} \quad \text{and} \quad \Lambda := \frac{1+r^*}{1-\theta} - (1-\theta) + \frac{(1+r^*)f'(\bar{y})h'(\bar{x})}{(1-\theta)\bar{p}^2 f''(\bar{y})}.$$

Note that $0 \leq \varpi < 1$. Eigenvalues λ and corresponding eigenvectors v are given by

$$\lambda_1 = (1-\theta) - \varpi, \quad v_1 = \begin{bmatrix} \frac{(1+r^*)f'(\bar{y})h'(\bar{x})}{(1-\theta)\bar{p}^2 f''(\bar{y})} + \varpi \\ \frac{(1+r^*)^2 f'(\bar{y})}{(1-\theta)\bar{p}^2 f''(\bar{y})} \end{bmatrix}$$

$$\lambda_2 = \frac{1+r^*}{1-\theta} + \frac{(1+r^*)f'(\bar{y})h'(\bar{x})}{(1-\theta)\bar{p}^2 f''(\bar{y})} + \varpi, \quad v_2 = \begin{bmatrix} -h''(\bar{x}) \\ \frac{(1+r^*)f'(\bar{y})h'(\bar{x})}{\bar{p}^2 f''(\bar{y})} + (1-\theta)\varpi \end{bmatrix}.$$

Since $\lambda_1\lambda_2 = 1 + r^* > 0$, $\lambda_2 > 1$ and $0 \leq \varpi < 1$, it follows that $0 < \lambda_1 < 1$. Hence, ϕ is a saddle, and v_1 and v_2 are the stable and unstable eigenspace. The mapping ϕ is trivially smooth and invertible, so it is a diffeomorphism. The center manifold theorem (e.g. Lines [12]) ensures that there exists a locally stable manifold $W^s_{loc}(\bar{z})$ around the steady state $\bar{z}$:

$$W^s_{loc}(\bar{z}) := \left\{ z \in \Theta | \lim_{n\to\infty} d[\phi^{\{n\}}(z), \bar{z}] = 0 \text{ and } \phi^{\{n\}}(z) \in \Theta \; \forall n \geq 0 \right\}.$$

The theorem states that $W^s_{loc}(\bar{z})$ forms a curve tangent to v_1. Since v_1 has a negative slope in the plane (p, x), $W^s_{loc}(\bar{z})$ is a curve in Θ where p is decreasing in x.

The global stable manifold (e.g. Galor [7]) of ϕ can be obtained as

$$W^s(\bar{z}) := \bigcup_{n\in\mathbb{N}} \left\{ \phi^{-1\{n\}}\left(W^s_{loc}(\bar{z})\right) \right\}$$

where

$$\phi^{-1}(z_{t+1}) = \begin{bmatrix} \phi_1^{-1}(z_{t+1}) \\ \phi_2^{-1}(z_{t+1}) \end{bmatrix} = \begin{bmatrix} p_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{1-\theta}{1+r^*} p_{t+1} + \frac{h(x_t)}{1-\theta} \\ \frac{x_{t+1}}{1-\theta} - \frac{\Gamma p^{\Psi}_{t+1}}{1-\theta} \end{bmatrix}.$$

Let $\Delta p_{t+1} := p_{t+1} - p_t$ and $\Delta x_{t+1} = x_{t+1} - x_t$. Since $\frac{\partial \Delta p_{t+1}}{\partial x_t} > 0$ and $\frac{\partial \Delta x_{t+1}}{\partial p_t} > 0$, we obtain a phase diagram as shown in Figure 5.

Since W^s is obtained by iterating ϕ^{-1}, W^s forms a unique, continuous and strictly decreasing function in $x \in \mathbb{R}_{++}$, and $\lim_{t\to\infty} z_t = \bar{z}$ for all $z_t \in W^s$ as shown in Figure 5. □

Proof of Theorem 3.2. Since ϕ is a saddle, any trajectory will eventually violate $(p_t, x_t) \geq 0$ if $p_0 < \rho(x_0)$. If $p_0 > \rho(x_0)$, $(p_t, x_t) \to \infty$. Hence, for any $\delta > 0$

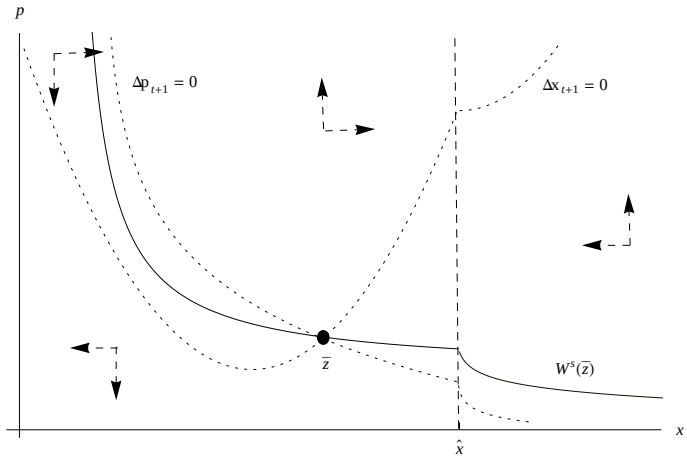


Figure 5: Phase diagram

there exists $N > 0$ such that $\frac{w}{x_t} < \delta$ and $p_N > \frac{\delta(1+r^*)}{r^*+\theta}$ when $t > N$. Since $p_{t+1} = \frac{1+r^*}{1-\theta}\left(p_t - \frac{w}{x_t}\right)$, $p_{t+1} > \frac{1+r^*}{1-\theta}(p_t - \delta)$. By iterating the equation we obtain $p_{t+N} > \left(\frac{1+r^*}{1-\theta}\right)^t \left(p_N - \frac{\delta(1+r^*)}{r^*+\theta}\right) + \frac{\delta(1+r^*)}{r^*+\theta}$.

Taking the limit

$$\begin{aligned}
&\lim_{k\to\infty} \left(\frac{1-\theta}{1+r^*}\right)^k p_{t+k} \\
&> \lim_{k\to\infty} \left(\frac{1-\theta}{1+r^*}\right)^N \left(\frac{1-\theta}{1+r^*}\right)^{k-N} \left[\left(\frac{1+r^*}{1-\theta}\right)^{t+k-N} \left(p_N - \frac{\delta(1+r^*)}{r^*+\theta}\right) + \frac{\delta(1+r^*)}{r^*+\theta}\right] \\
&= \lim_{k\to\infty} \left(\frac{1-\theta}{1+r^*}\right)^N \left[\left(\frac{1+r^*}{1-\theta}\right)^t \left(p_N - \frac{\delta(1+r^*)}{r^*+\theta}\right) + \frac{\delta(1+r^*)}{r^*+\theta}\left(\frac{1-\theta}{1+r^*}\right)^{k-N}\right] \\
&> 0
\end{aligned}$$

□

Proof of Lemma 4.1. It is trivial from (12) that $1 + r_{j,t+1} > 1 + r^*$ if and only if $(1 + r_{j,t+1})b_{j,t} > C_j$. If $(1 + r_{j,t+1})b_{j,t} > C_j$, then the optimal debt contract is given by $(1 - q)(1 + r_{j,t+1}) + qC_j/b_{j,t} = 1 + r^*$ Since $1 + r^*$ is a convex combination of $1 + r_{j,t+1}$ and $C_j/b_{j,t}$, it follows that $1 + r_{j,t+1} > 1 + r^*$ if and only if $C_j/b_{j,t} < 1 + r^*$. □

Proof of Proposition 4.1. From Theorem 3.1, ϕ is a saddle. Let $\acute{\rho}(x)$ and $\grave{\rho}(x)$ denote the price functions satisfying $\Delta p_{t+1} = 0$ and $\Delta x_{t+1} = 0$ respectively. Note that the phase diagram of φ is the same as the one of ϕ in Figure 5 except that $\Delta p_{t+1} = 0$

locus pivots around $\bar{z}$ in the clockwise fashion. Moreover, the dynamics of ϕ and φ are topologically equivalent. Hence,

$$\hat{\rho}(x) = \begin{cases} \acute{\rho}(x) & \text{for } x \in (0, \bar{x}) \\ \grave{\rho}(x) & \text{for } x \in [\bar{x}, \infty) \end{cases}$$

Figure 5 shows the properties in the proposition. □

Proof of Lemma 5.1. Under the fundamental price dynamics, contractors borrow no more than their pledgeable income $\rho(x_{t+1})f(y_t)$. By construction, this implies $l_1(x) > \rho(x)$. The threshold price $p_t = l_1(x_t)$ is defined by a pair (p_t, x_t) that satisfies

$$(1 + r^*)(y_t - w) = \rho(x_{t+1})f(y_t) \Leftrightarrow (1 + r^*)\left(1 - \frac{w}{y_t}\right) = \frac{\rho(x_{t+1})f(y_t)}{y_t}.$$

We observe from (8) and (11) that x_{t+1} is strictly increasing in both p_t and x_t and from (7) and (11) that y_t is also strictly increasing in both p_t and x_t. According to Assumption 1, $\frac{f(k)}{k}$ is decreasing. Hence, $l_1(x)$ is strictly decreasing.

Next, the threshold price $p_t = l_2(x_t)$ is defined by a pair (p_t, x_t) that satisfies

$$(1 + r^*)(p_t x_t - w) = (1 - \theta)\rho(x_{t+1})x_t.$$

For $x < \hat{x}$, the credit constraint is not binding. Hence, $l_2(x_t) \geq \rho(x)$. Since producers are credit-constrained for $x > \hat{x}$, it must be that $l_2(x) = \rho(x)$. Next, we observe from (8) and (11) that x_{t+1} is strictly increasing in both p_t and x_t. Hence, $l_2(x)$ is strictly decreasing. □

Sketch of the proof of Proposition 6.1. The idea is to show that the fundamental value constraint (FVC) restricts the space (x, p) in such a way that any trajectory with exploding x and p will be ruled out. With the FVC, banks do not lend more than the future fundamental value of the collateral. In other words, the credit limit is the minimum between the future value and the fundamental value of the collateral:

$$\begin{aligned} (1 + r^*)(y_t - w) &= \min\{\rho(x_{t+1}), p_{t+1}\}f(y_t) \\ (1 + r^*)(p_t x_t - w) &= \min\{\rho(x_{t+1}), p_{t+1}\}(1 - \theta)x_t. \end{aligned} \tag{15}$$

If both constraints bind, then (8) and (15) define a price function $p_f(x)$ where $p_f'(x) < 0$ and $p_f(x) > \rho(x)$. This implies that the domain of the dynamical

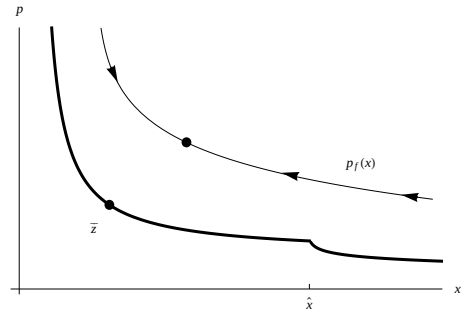


Figure 6: Binding fundamental value constraints

system is restricted to be below $p_f(x)$, and thus any bubbly dynamics where $(x_t, p_t) \to \infty$ cannot be a rational expectations equilibrium. For $p_t > \rho(x_t)$, the system without the policy shows that $\rho(x_{t+1}) < p_{t+1}$ and we can show that only paths that converge to some steady states are feasible. For $p_t < \rho(x_t)$, the system without the policy shows that $p_{t+1} < \rho(x_{t+1})$ and thus the dynamics in Theorem 3.1 hold which will violate the non-negativity condition. We conclude that (15) rules out all explosive dynamics, and therefore bubbles. Trivially, the fundamental price is not affected by (15). □

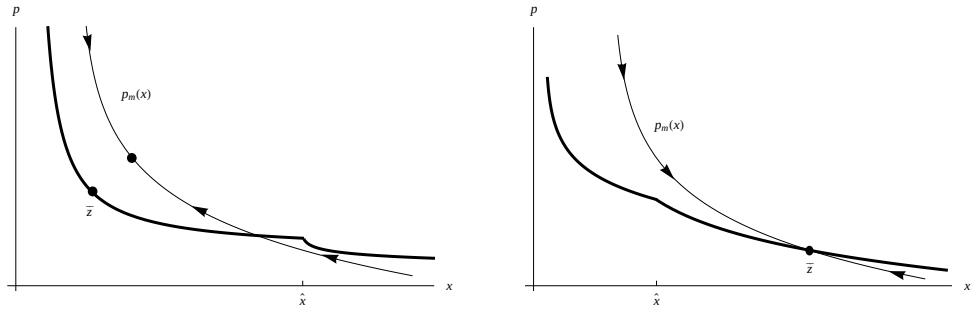
Sketch of the proof of Proposition 6.2. The idea is similar to the proof of Proposition 6.1. The marked-to-market-value constraint (MMVC) for contractors and producers can be written as[20]

$$(1 + r^*)(y_t - w) \leq p_t f(y_t) \tag{16}$$

$$(1 + r^*)(p_t x_t - w) \leq p_t(1 - \theta)x_t \tag{17}$$

Equation (17) defines a price function $p_m(x)$ where $p_m'(x) < 0$, $\lim_{x\to 0} p_m(x) = \infty$, and $\lim_{x\to\infty} p_m(x) = 0$. If $\bar{x} > \hat{x}$, then the future value constraint is binding for contractors in the steady state. Since $p_t = p_{t+1}$ in the steady state, the steady states where the future value constraint and the MMVC are binding for contractors must coincide. This implies that $p_m(x)$ crosses $\rho(x)$ from above at $\bar{x}$. If $\bar{x} < \hat{x}$, then $p_m(x) < \rho(x)$ for $x > \hat{x}$ and $p_m(x) > \rho(x)$ for $x < \bar{x}$. We obtain two generic cases: $\bar{x} < \hat{x}$ and $\bar{x} > \hat{x}$ as shown in Figure 7.

[20]This constraint for producers can also be interpreted as a particular margin constraint: $b_{t,1} = \lambda p_t x_t$ where $\lambda = \frac{1-\theta}{1+r^*}$.



(a) $\bar{x} < \hat{x}$ (b) $\bar{x} > \hat{x}$

Figure 7: Binding MMVC for producers

This implies that the domain of the dynamical system is restricted to be below $p(x)$, and thus any bubbly dynamics where $(x_t, p_t) \to \infty$ cannot be a rational expectations equilibrium. We can show that the only feasible path is the equilibrium converging to the steady state on $p_m(x)$. We conclude that (17) rules out all explosive dynamics, and therefore bubbles.

Lastly, it needs to be highlighted that although parts of the fundamental price path $p = \rho(x)$ are constrained by (17), the original steady state $(\bar{x}, \bar{p})$ survives. The steady state where the MMVC is binding is not lower than the original one. In other words, the equilibrium dynamics in the presence of MMVC may converge to some steady states with price levels not lower than the fundamental price level. □

Proof of Lemma 7.1. We first examine the growth rate of the expenditure on bubbles in the bubbly equilibrium. If $p_0 > \rho(x_0)$, Theorem 3.1 implies that $(p_t, x_t) \to \infty$. From (11) the limit of the price growth rate is given by

$$\lim_{t\to\infty} \frac{p_{t+1}}{p_t} = \frac{1+r^*}{1-\theta}\left(1 - \lim_{t\to\infty} \frac{w}{p_t x_t}\right) = \frac{1+r^*}{1-\theta} \tag{18}$$

Let $\Psi := \frac{\alpha}{1-\alpha}$, $\Gamma := (\frac{\alpha}{1+r^*})$ and $m_t := \frac{x_t}{p_t^\Psi}$. From (8)

$$\begin{aligned} x_{t+1} &= (1-\theta)x_t + \Gamma p_{t+1}^\Psi \\ \frac{x_{t+1}}{p_{t+1}^\Psi} &= (1-\theta)\frac{x_t}{p_t^\Psi}\frac{p_t^\Psi}{p_{t+1}^\Psi} + \Gamma \\ m_{t+1} &= (1-\theta)\left(\frac{p_t}{p_{t+1}}\right)^\Psi m_t + \Gamma. \end{aligned}$$

Since $\lim_{t\to\infty} \frac{p_{t+1}}{p_t} = \frac{1+r^*}{1-\theta}$ and $(\frac{1+r^*}{1-\theta})^\Psi > (1-\theta)$, there exists a finite $\tilde{t}$ such that $(1-\theta)(\frac{p_t}{p_{t+1}})^\Psi < 1$ for all $t \geq \tilde{t}$. Hence, it follows that $0 < \lim_{t\to\infty} m_t < \infty$. The limit of the stock growth rate is given by

$$\begin{aligned}
\lim_{t\to\infty} \frac{x_{t+1}}{x_t} &= \lim_{t\to\infty} \frac{m_{t+1}}{m_t} \left(\frac{p_{t+1}}{p_t}\right)^\Psi \\
&= \lim_{t\to\infty} \left[(1-\theta)\left(\frac{p_t}{p_{t+1}}\right)^\Psi + \frac{\Gamma}{m_t}\right] \left(\frac{p_{t+1}}{p_t}\right)^\Psi \\
&= 1-\theta+\Gamma\left(\frac{1+r^*}{1-\theta}\right)^\Psi \lim_{t\to\infty} \frac{1}{m_t}. \qquad (19)
\end{aligned}$$

From (18) and (19) the growth rate of the expenditure on bubbles is now given by

$$\begin{aligned}
\lim_{t\to\infty} \tfrac{p_{t+1}x_{t+1}}{p_t x_t} &= \lim_{t\to\infty} \tfrac{p_{t+1}}{p_t} \lim_{t\to\infty} \tfrac{x_{t+1}}{x_t} \\
&= \tfrac{1+r^*}{1-\theta}\left[1-\theta+\Gamma\left(\tfrac{1+r^*}{1-\theta}\right)^\Psi \lim_{t\to\infty} \tfrac{1}{m_t}\right] \\
&= 1+r^*+\Gamma\left(\tfrac{1+r^*}{1-\theta}\right)^{\Psi+1} \lim_{t\to\infty} \tfrac{1}{m_t} \\
&> 1+r^*.
\end{aligned}$$

We now examine the growth rate of the expenditure on bubbles in the sunspot equilibrium. If $p_0 > \rho(x_0)$, Proposition 4.1 states that $(p_t, x_t) \to \infty$. The price dynamics in the sunspot equilibrium is given by

$$p_{t+1} = \frac{1+r^*}{(1-q)(1-\theta)}\left(p_t - \frac{w}{x_t}\right) - \frac{q\rho(x_{t+1})}{(1-q)}.$$

We obtain the limit of the price growth rate

$$\begin{aligned}
\lim_{t\to\infty} \frac{p_{t+1}}{p_t} &= \frac{1+r^*}{(1-q)(1-\theta)}\left(1-\lim_{t\to\infty}\frac{w}{p_t x_t}\right) - \lim_{t\to\infty}\frac{q\rho(x_{t+1})}{(1-q)p_t} \\
&= \frac{1+r^*}{(1-q)(1-\theta)} \qquad (20)
\end{aligned}$$

and the limit of the expected price growth rate

$$\lim_{t\to\infty} \frac{\mathbb{E}_t(p_{t+1})}{p_t} = \frac{1+r^*}{1-\theta}. \qquad (21)$$

Similarly as in the bubbly equilibrium

$$\begin{aligned}
x_{t+1} &= (1-\theta)x_t + \Gamma\left(\mathbb{E}_t(p_{t+1})\right)^\Psi \\
m_{t+1} &= (1-\theta)\left(\tfrac{p_t}{p_{t+1}}\right)^\Psi m_t + \Gamma\left(\tfrac{\mathbb{E}_t(p_{t+1})}{p_{t+1}}\right)^\Psi \qquad (22) \\
\tfrac{m_{t+1}}{m_t} &= (1-\theta)\left(\tfrac{p_t}{p_{t+1}}\right)^\Psi + \tfrac{\Gamma}{m_t}\left(\tfrac{\mathbb{E}_t(p_{t+1})}{p_{t+1}}\right)^\Psi
\end{aligned}$$

where $\frac{\mathbb{E}_t(p_{t+1})}{p_{t+1}} < 1$. From (20) and (21) the dynamics in (22) implies

$$\lim_{t\to\infty} m_t > 0 \tag{23}$$

Using (21) and (22), we obtain the limit of the stock growth rate as

$$\begin{aligned}
\lim_{t\to\infty} \frac{x_{t+1}}{x_t} &= \lim_{t\to\infty} \frac{m_{t+1}}{m_t}\left(\frac{p_{t+1}}{p_t}\right)^{\Psi} \\
&= \lim_{t\to\infty}\left[(1-\theta)\left(\frac{p_t}{p_{t+1}}\right)^{\Psi} + \frac{\Gamma}{m_t}\left(\frac{\mathbb{E}_t(p_{t+1})}{p_{t+1}}\right)^{\Psi}\right]\left(\frac{p_{t+1}}{p_t}\right)^{\Psi} \\
&= 1-\theta+\lim_{t\to\infty}\frac{\Gamma}{m_t}\left(\frac{\mathbb{E}_t(p_{t+1})}{p_t}\right)^{\Psi} \\
&= 1-\theta+\Gamma\left(\frac{1+r^*}{1-\theta}\right)^{\Psi}\lim_{t\to\infty}\frac{1}{m_t}.
\end{aligned}$$

Finally using (23) we obtain the growth rate of the expenditure on bubbles as

$$\begin{aligned}
&\mathbb{E}_t\left(\lim_{t\to\infty}\frac{p_{t+1}x_{t+1}}{p_t x_t}\right) \\
&= \lim_{t\to\infty}(1-q)^{t+1}\lim_{t\to\infty}\frac{p_{t+1}x_{t+1}}{p_t x_t} + \sum_{k=1}^{\infty} q(1-q)^{k-1}\lim_{t\to\infty}\frac{p_{t+1}x_{t+1}}{p_t x_t}\Big|_{T=k} \\
&= \lim_{t\to\infty}(1-q)^{t+1}\lim_{t\to\infty}\frac{p_{t+1}}{p_t}\lim_{t\to\infty}\frac{x_{t+1}}{x_t}+1 \\
&= \lim_{t\to\infty}(1-q)^{t+1}\frac{1+r^*}{(1-q)(1-\theta)}\left[1-\theta+\Gamma\left(\frac{1+r^*}{1-\theta}\right)^{\Psi}\lim_{t\to\infty}\frac{1}{m_t}\right]+1 \\
&= \lim_{t\to\infty}(1-q)^{t+1}\left[\frac{1+r^*}{1-q}+\frac{\Gamma}{1-q}\left(\frac{1+r^*}{1-\theta}\right)^{\Psi+1}\lim_{t\to\infty}\frac{1}{m_t}\right]+1 \\
&= 1
\end{aligned}$$

□

References

[1] R. S. Aiyagari, Uninsured Idiosyncratic Risk and Aggregate Saving, Quarterly Journal of Economics, 109(3) (1994) 659-684.

[2] F. Allen and D. Gale. Bubbles and Crises, The Economic Journal, 110(460) (2000) 236-255.

[3] R. J. Caballero and A. Krishnamurthy, Bubbles and Capital Flow Volatility: Causes and Risk Management, Journal of Monetary Economics, 53(1) (2006) 35-53.

[4] R. J. Caballero, E. Farhi and M. L. Hammour, Speculative Growth: Hints from the U.S. Economy, The American Economic Review, 96(4) (2006) 1159-1192.

[5] E. Challe and X. Ragot, Bubbles and Self-Fulfilling Crises, The B.E. Journal of Macroeconomics, 11(1) (2011) 1935-1690.

[6] E. Farhi and J. Tirole, Bubbly Liquidity, Review of Economic Studies, 79(2) (2012) 678-706.

[7] O. Galor, Discrete Dynamical Systems, Springer (2007).

[8] T. Hirano and N. Yanagawa, Asset Bubbles, Endogenous Growth, and Financial Frictions, mimeo, Univeristy of Tokyo (2013).

[9] N. Kiyotaki and J. Moore, Credit Cycles, Journal of Political Economy, 105(2) (1997) 211-248.

[10] N. R. Kocherlakota, Bursting Bubbles: Consequences and Cures, mimeo, Univerisity of Minnesota (2009).

[11] P. Krugman, Balance Sheets, the Transfer Problem, and Financial Crises, International Tax and Public Finance, 6(4) (1999) 459-472.

[12] M. Lines, Nonlinear Dynamical Systems in Economics, Springer (2005).

[13] A. Mian and A. Sufi, Household Leverage and the Recession of 2007-09, IMF Economic Review, 58(1) (2010) 74-116.

[14] A. Mian and A. Sufi, The Consequences of Mortgage Credit Expansion: Evidence From the U.S. Mortgage Default Crisis, Quarterly Journal of Economics, 123(4) (2009) 1449-96.

[15] A. Martin and J. Ventura, Theoretical Notes on Bubbles and the Current Crisis, IMF Economic Review, 59(1) (2011) 6-40.

[16] A. Martin and J. Ventura, Economic Growth with Bubbles, American Economic Review, 102(6) (2012) 3033-3058.

[17] J. Olivier, Growth-enhancing bubbles, International Economic Review, 41 (2000) 133-151.

[18] M. S. Santos and M. Woodford, Econometrica, 65(1) (1997) 19-57.

[19] J. Tirole, Asset Bubbles and Overlapping Generations, Econometrica, 53(6) (1985) 1499-1528.

[20] J. Ventura, Bubbles and Capital Flows, Journal of Economic Theory, 147(2), (2012) 738-758.

[21] P. Weil, Confidence and the Real Value of Money in an Overlapping Generations Economy, Quarterly Journal of Economics, 102(1) (1987) 122.