Comput Econ (2008) 32:163–181
DOI 10.1007/s10614-008-9137-3

Asset Pricing and Productivity Growth: The Role of Consumption Scenarios

Volker Böhm · Tomoo Kikuchi · George Vachadze

Accepted: 27 March 2008 / Published online: 26 April 2008

Abstract The paper analyzes the performance of asset prices implied by an aggregate macroeconomic growth model under two different consumption hypotheses: overlapping generations of agents with two period lives versus the infinitely lived agent. The production side of the economy is described by a random growth model with a competitive labor market and an exogenously given random dividend payout ratio. For an isoelastic technology with multiplicative production shocks this implies a random dynamical system for the firm's rate of profit with a unique asymptotically stable random fixed point for a large class of productivity growth and dividend payout ratio processes. Based on an extensive numerical study of stationary solutions we show that the two consumption scenarios imply a limited number of diverse effects regarding equity and bond returns and equity premia.

Keywords Asset pricing · Computational and simulation techniques · Economic growth · Equity premium · Portfolio choice

JEL classification C15 · C63 · G11 · G12 · E21 · E44 · F43

V. Böhm (✉) · G. Vachadze
Department of Economics, Bielefeld University, Postfach 100 131, 33501 Bielefeld, Germany
e-mail: vboehm@wiwi.uni-bielefeld.de

G. Vachadze
e-mail: gvachadze@wiwi.uni-bielefeld.de

T. Kikuchi
Department of Economics, National University of Singapore,
1 Arts Link, AS2-04-30, Singapore 117570, Singapore
e-mail: ecstk@nus.edu.sg

1 Introduction

The two standard models used in the real business cycle literature are the model with overlapping generations of consumers (OLG) and the one with an infinitely lived agent (ILA). While the OLG model has an explicit demographic structure to describe key life-cycle stages, the ILA model treats population as a single, infinitely lived household. For a comparative analysis of these two approaches with respect to their performance and explanatory power of asset returns, two aspects which are treated differently in the two models seem to be of importance: the bequest motive and the life-cycle structure. Under the bequest motive, which is central to the planner-based ILA framework, allocation decisions are made by an infinitely lived agent acting on behalf of both the present and all future generations. Thus, an action by the agent affects the entire infinite life time. In contrast, given the demographic structure of an OLG model, actions of different age groups have consequences on consumers beyond their own life time. When bequests are incorporated into an OLG model the differences of these effects become small. In fact, in some cases the two models perform similarly (see Barro 1974; Blanchard 1985; Blanchard and Fischer 1989). When demographic heterogeneity is introduced into the OLG framework, at any one time, allocation decisions are made simultaneously by agents of different age groups. Such life-cycle considerations cause different generations to face different incomes as well as different risks concerning future consumption. As a consequence they value equity and bond investments differently. Therefore, one would expect that the two scenarios should perform differently when intertemporal allocations are supported by asset markets.

A number of recent studies in the asset pricing literature argue that the life-cycle consideration present in OLG economies can play a crucial role in improving the performance of consumption based asset pricing models, for example Constantinides et al. (2002) and Storesletten et al. (2007). They obtain this result by introducing imperfections in the credit market or allowing persistent and idiosyncratic risk with counter cyclical volatility. Other implications of life-cycle considerations on asset prices have been studied by Ríos (1994, 1996), Huggett (1996), and Hillebrand and Wenzelburger (2006). These papers argue that individuals of different age groups may respond differently to aggregate macroeconomic shocks. While some of these papers assert a general advantage of the OLG framework over the infinitely lived optimizing agents model, a direct comparison of the performance of the two approaches has not been carried out.

The macroeconomic literature does not contain many comparative investigations of these two different approaches. One exception can be found in the literature on environmental economics where the issue of policy effectiveness is investigated under the two scenarios (see Müller-Fürstenberger and Predivoli 1997). The authors argue that the results from both models are sufficiently close to each other and that the insights provided can be treated equally in terms of the policy relevance. In contrast, Howarth (1996, 1998) finds substantial differences between the results from OLG and ILA models and concludes that the performance of the OLG model dominates since the ILA model lacks qualitatively important demographic features.

The challenge taken up in this paper is to provide a *comparative* evaluation of the asset pricing model under the two competing consumption scenarios: overlapping generations with two period lives versus the infinitely lived agent. We restrict our analysis to the case with two period lives which minimizes the heterogeneity of the demographic structure. The main findings of the paper can be summarized as follows:

- Risk premia implied by the OLG model are higher than those implied by the ILA model for reasonable parameter configurations. However, the difference in risk premium prediction is statistically insignificant after taking the volatility of risk premia into consideration.
- Equity and bond returns implied by the ILA model are higher than those implied by the OLG model for reasonable values of parameters. Equity returns predicted by the OLG model fall in a reasonable range for reasonable values of the parameter of relative risk aversion. However, equity returns are overpredicted by the ILA model as long as the parameter of relative risk aversion exceeds two. Both models overpredict the bond return.
- Equity returns implied by both models are highly correlated with productivity growth when the relative risk aversion is low, making realized productivity growth one of the most important predictors for equity returns. As risk aversion increases the importance of productivity growth in predicting the equity market weakens, while the importance of the capital growth rate increases. However, the realized dividend payout ratio is one of the most important predictors for bond returns.

Section 2 presents the dynamic model and describes the productivity growth and capital accumulation process. After showing existence and uniqueness of a stable random fixed point, we describe economies with two different demographic structures. Section 3 contains the comparative discussion of these two models and provides the intuition as to why and when either model can display a better performance. Finally, Sect. 4 summarizes the results and concludes.

2 The Model

Consider a market economy evolving in discrete time with a consumption sector, a production sector, and four markets operating in every period: a market for a single produced commodity usable for consumption and investment, a market for labor, *plus* markets for equity/shares, and a discount bond. The discount bond pays one unit of consumption goods in the subsequent period while the shares pay random dividends. For simplicity it is assumed that the total supply of shares is constant and normalized to unity while the aggregate net supply of the bond is normalized to zero. All markets in the economy operate under perfect competition implying price taking behavior by all agents.

2.1 The Production Sector

The production sector of the economy is described by a single infinitely lived firm producing a homogeneous output by using a standard neoclassical technology with capital and labor as inputs. Labor supply is fixed and normalized to unity. Therefore, we can write the production function in per capita terms as $y_t = f(A_t, k_t) := A_t k_t^{\theta}$, where A_t is the Hick's neutral aggregate productivity level, k_t is the capital labor ratio, and $\theta \in (0, 1)$ is the share of capital in production. Output in each period is subject to a random productivity shock. The evolution of aggregate productivity is described by the equation $A_t = A_{t-1}e^{a_t}$, where $a_t = \mu_a + \sigma_a \varepsilon_{1t}$ is the rate of productivity growth[1] with mean μ_a and volatility σ_a. After the productivity shock occurs, the firm pays wages according to the marginal product rule implying $w_t = w(A_t, k_t) := f(A_t, k_t) - k_t f_k(A_t, k_t)$. A part $\eta_t \in [0, 1]$ of the remaining profit $\pi_t = \pi(A_t, k_t) := f(A_t, k_t) - w(A_t, k_t)$ is distributed as dividends to share holders while the rest is invested in physical capital. Therefore, dividend payments and physical capital investment in any period are given by $d_t = d(A_t, k_t, \eta_t) := \eta_t \pi(A_t, k_t)$ and $i_t = i(A_t, k_t, \eta_t) := (1 - \eta_t)\pi(A_t, k_t)$.

The evolution of the firm's dividend payout ratio η_t is stochastic and described by the equation $\eta_t = \min\big(1, \max\big(0, \mu_\eta + \sigma_\eta \varepsilon_{2t}\big)\big)$, where μ_η and σ_η describe the mean and the volatility of the dividend payout ratio. This type of dividend payout policy can be justified if firms use an adaptive process minimizing a quadratic loss function for the deviation from an optimal target dividend. The assumption is also consistent with the findings of a classical empirical study by Lintner (1956), in which annual dividend payments follow the partial adjustment model. Since in our model one period corresponds to about 35 years during which full adjustment happens, the dividend payment model is reduced to this static one. We assume that for any $t = 1, 2, \ldots$, the pair of shocks $(\varepsilon_{1t}, \varepsilon_{2t})$ has a bivariate normal distribution with $\mathbb{E}\varepsilon_{1t} = \mathbb{E}\varepsilon_{2t} = 0$, $\mathbb{V}\varepsilon_{1t} = \mathbb{V}\varepsilon_{2t} = 1$ and $corr(\varepsilon_{1t}, \varepsilon_{2t}) = \rho$.

Given a constant rate of capital depreciation $\delta \in [0, 1]$ and a triple (η_t, k_t, π_t) of the current dividend payout ratio, capital stock, and profit, the level of physical capital in the next period is determined by

$$k_{t+1} = (1 - \delta)k_t + (1 - \eta_t)\pi_t. \tag{1}$$

Let $g_{t+1} := k_{t+1}/k_t$ denote the growth factor of capital (one plus the growth rate). Then, Eq. 1 implies that

$$g_{t+1} = (1 - \delta) + (1 - \eta_t)\xi_t, \tag{2}$$

where $\xi_t := \pi_t/k_t$ is the rate of profit, i.e. profit per unit of capital. Since the firm's profit in any period is given by $\pi_t = A_t \theta k_t^\theta$, it follows that

$$\xi_{t+1} = \frac{\pi_{t+1}}{k_{t+1}} = \frac{\pi_{t+1}}{\pi_t}\frac{\pi_t}{k_t}\frac{k_t}{k_{t+1}} = e^{a_{t+1}}\xi_t g_{t+1}^{\theta-1}. \tag{3}$$

Substituting g_{t+1} from Eq. 2 into Eq. 3 we obtain an explicit non-linear difference equation

$$\xi_{t+1} = G(\xi_t, a_{t+1}, \eta_t) := e^{a_{t+1}}\xi_t \left(1 - \delta + (1 - \eta_t)\xi_t\right)^{\theta-1}, \tag{4}$$

[1] Prescott (1986) uses the same assumption when log productivity follows a random walk.

describing the evolution of ξ_t. Therefore, the set of equations

$$\begin{aligned} a_{t+1} &= \mu_a + \sigma_a \varepsilon_{1t+1}, \\ \eta_{t+1} &= \min\left(1, \max\left(0, \mu_\eta + \sigma_\eta \varepsilon_{2t+1}\right)\right), \\ \xi_{t+1} &= G(\xi_t, a_{t+1}, \eta_t), \\ g_{t+1} &= (1-\delta) + (1-\eta_t)\xi_t, \\ A_{t+1} &= A_t e^{a_{t+1}}, \\ k_{t+1} &= k_t g_{t+1}, \end{aligned} \tag{5}$$

with initial conditions $\eta_0 = \overline{\eta}$, $\xi_0 = \overline{\xi}$, $k_0 = \overline{k}$, and $A_0 = \overline{A}$, describes the evolution of the production side of the economy. Since $\mathbb{E} \log a_t < \infty$, $\mathbb{E} \log \eta_t < \infty$, and the rate of profit is described by a random family of strictly monotonically increasing and strictly concave maps

$$G(\cdot, a_{t+1}, \eta_t) \colon \mathbb{R}_+ \to \mathbb{R}, \text{ for any } a_{t+1} \in \mathbb{R}, \quad \eta_t \in [0, 1], \tag{6}$$

there exists an associated representation as a random dynamical system. This allows a full *dynamic* analysis in a state space representation of random growth and accumulation, inducing the possibility of a complete *statistical* analysis of the associated process.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the canonical representation of the stochastic process $\{a_t, \eta_t\}$ and $\vartheta \colon \Omega \to \Omega$ denote the left shift for any $\omega \in \Omega$. Then, $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta^t))$ together with the mapping G becomes a random dynamical system in the sense of Arnold (1998), allowing the application of some of the powerful results of that theory.

The mapping G has two fixed points $\bar{\xi}_1 = 0$ and $\bar{\xi}_2 = \frac{1}{1-\eta}(e^{\frac{a}{1-\theta}} - (1-\delta))$ with $\bar{\xi}_1 \leq \bar{\xi}_2$ if and only if $a \geq (1-\theta)\ln(1-\delta)$. Hence, $X := [0, \infty) \subset \mathbb{R}_+$ is a forward invariant set and $G(\cdot, a, \eta) \colon X \to X$ is a family of contractions. Therefore, it follows from the results by Schmalfuss (1996, 1998) that the long run development of the rate of profit ξ_t is described by a unique asymptotically stable random fixed point,[2] i.e. a unique stationary random variable $\xi^* \colon \Omega \to \mathbb{R}_+$, satisfying $\xi^*(\vartheta\omega) = G(\xi^*(\omega), a(\vartheta\omega), \eta(\omega))$, almost surely. In addition, the random fixed point is asymptotically stable, i.e. on some set $\mathcal{U} \subset \Omega \times X$, $\lim_{t\to\infty} ||\xi_t(\omega) - \xi^*(\vartheta^t\omega)|| = 0$ for all $(\omega, \xi_0(\omega)) \in \mathcal{U}$. In other words, almost all orbits converge point wise to the unique stationary solution and the limiting distribution can be obtained from time averages.

2.1.1 Growth of Productivity and Capital in the Long Run

Existence and uniqueness of an asymptotically stable random fixed point combined with ergodicity imply that the invariant behavior of the random vector $(a_t, \eta_t, \xi_t, g_t)$ is characterized completely by the limiting statistical properties of a time series

[2] See Böhm and Wenzelburger (2002) for details.

Table 1 Standard parameter set

δ	θ	μ_a	σ_a	μ_η	σ_η	ρ
0.95	0.33	1.06	0.07	0.30	0.06	0.00

(solution) of a *single sample path* of the perturbation $\{(\varepsilon_{1t}, \varepsilon_{2t})\}$. Therefore, for a comparative analysis with different values of the parameters, it is sufficient to consider a single time series for each set of values of the parameters using the *same* sample path of the noise.

Since we consider an OLG model with only two period lived consumers we assume one period to correspond to about 35 years. Therefore, to obtain values close to empirical estimates, we make the following adjustments of the parameter values. We assume 8% annual depreciation of physical capital implying a rate of depreciation over 35 years $\delta = 1 - (1 - 0.08)^{35} \approx 95\%$ and the share of capital in production to be $\theta = 33\%$. The mean and the volatility of annual productivity growth are taken to be 3% and 1.2%, respectively. For 35 years, this implies a mean $\mu_a = 35 \times 3\% = 105\%$ and a volatility $\sigma_a = \sqrt{35} \times 1.2\% = 7\%$. The mean and the volatility of the dividend payout ratio over 35 years are taken to be 30% and 6%, respectively. The correlation coefficient ρ is set to be zero. Later we will check the robustness of the results by varying its value. The parameters given in Table 1 serve as a benchmark case unless it is otherwise indicated.

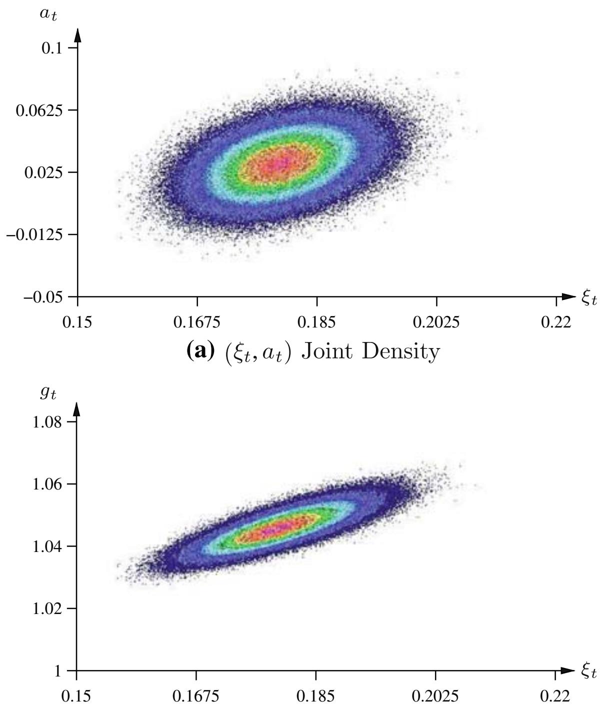
For the parameter values in Table 1, we generated the time series of the random dynamical system (5) using a *given* sample path of the noise $\{(\varepsilon_{1t}, \varepsilon_{2t})\}$ of length 10^6. Figure 1a displays the joint density function[3] of the annualized rate of profit and the annualized productivity growth. Figure 1b displays the joint density function of the annualized rate of profit and the annualized capital growth. The color coding of the support of the invariant distribution provides information of the relative frequencies (histogram) corresponding to that of a physical topographical map, brown/red/light gray = high frequencies and green/blue/dark gray = low frequencies. Table 2 displays the moments of the sample path $\{(a_t, \eta_t, \xi_t, g_t)\}$. As the table indicates, for given parameter values, the implied mean annualized capital growth and the mean annualized rate of profit are 4.6%[4]and 18%[5], respectively. Table 3 shows the sample correlations of the four random variables revealing high values $corr(a_t, \xi_t) = 39\%$ and $corr(\xi_t, g_t) = 79\%$, while correlations between other variables are insignificant or non-existent.

For an evaluation of the role of cross correlations for our investigation of asset returns, it is useful to analyze the interaction of the two noise processes and their impact on the cross correlation between the growth rates of dividends and of wages. Our model implies that, due to the volatility of the dividend payout ratio, the correlation between

[3] All simulations are carried out using the software package MACRODYN specifically designed for the simulation of deterministic and stochastic dynamical systems; see Böhm (2003).

[4] Average annualized capital growth rate in the US during the period 1970–1985 was 2.83%.

[5] This value of the rate of profit implies that the ratio of output to capital is 54%, while Campbell (1994) reports 47% for the same quantity.



(a) (ξ_t, a_t) Joint Density

(b) (ξ_t, g_t) Joint Density

Fig. 1 Invariant behavior of the production sector of the economy

the growth rates of dividends and of wages are not perfect.[6] In order to understand the role of the parameter ρ on these cross effects, let

$$\tau^w_{t+1} := \frac{w_{t+1}}{w_t}, \text{ and } \tau^d_{t+1} := \frac{d_{t+1}}{d_t}, \tag{7}$$

denote the growth rates of wages and dividends. Then

$$\tau^d_{t+1} = \frac{d_{t+1}}{d_t} = \frac{d_{t+1}}{w_{t+1}} \frac{w_{t+1}}{w_t} \frac{w_t}{d_t} = \frac{\eta_{t+1}}{\eta_t} \tau^w_{t+1}, \tag{8}$$

[6] The magnitude of this correlation, however, can play a decisive role for the performance of risk premia implied by the two alternative models. When the correlation is sufficiently high the risk premium implied by the ILA model can be higher than the one implied by the OLG model. The reason for this is that the risk averse infinitely lived agent, who receives wage and dividend income in each period, demands a higher risk premium for his investment in the equity market. In contrast, old agents in the OLG model do not face wage income risk and, therefore, are willing to accept a lower risk premium for equity investment.

Table 2 Sample moments

	Mean	SD	Skewness	Kurtosis
a_t	0.030	0.012	0.003	−0.014
η_t	0.300	0.010	0.040	−0.017
ξ_t	0.180	0.005	0.110	−0.015
g_t	1.046	0.004	0.078	0.002

Table 3 Sample correlation

	a_t	η_t	ξ_t	g_t
a_t	1.00			
η_t	0.00	1.00		
ξ_t	0.39	0.00	1.00	
g_t	−0.02	0.00	0.79	1.00

where

$$\tau_{t+1}^{w} = \frac{w_{t+1}}{w_t} = \frac{A_{t+1}(1-\theta)k_{t+1}^{\theta}}{A_t(1-\theta)k_t^{\theta}} = e^{a_{t+1}} g_{t+1}^{\theta}. \tag{9}$$

When the dividend payout ratio is constant, $\sigma_\eta = 0$ then $\eta_{t+1}/\eta_t = 1$ and $\tau_{t+1}^{w} = \tau_{t+1}^{d}$ implying a perfect correlation between the growth rates of wages and dividends. As the volatility of the dividend payout ratio increases, the correlation between τ_t^w and τ_t^d becomes smaller. This correlation can also be affected by a change of the value of the parameter ρ. Figure 2 displays the joint density functions of the growth rates of wages and dividends for different values of ρ. When $\rho = 0$ the value of this correlation is $corr(\tau_t^w, \tau_t^d) = 0.27$. In contrast, $corr(\tau_t^w, \tau_t^d) = -0.38$ when $\rho = -0.90$ and $corr(\tau_t^w, \tau_t^d) = 0.76$ when $\rho = 0.90$.

2.2 The Consumption Sector

The two scenarios for the consumption sector to be investigated are overlapping generations of consumers with two period lives and consumption in both periods *and* an infinitely lived agent optimizing and an infinite horizon. In both scenarios consumers have the same instantaneous utility and the same time discount. Instantaneous preferences are described by the CRRA utility function

$$u(c) := \begin{cases} \ln c & \text{if } \gamma = 1 \\ \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \end{cases} \tag{10}$$

and the time discount factor is $\beta \in (0, 1)$. Consumers choose optimal consumption plans buying equity shares and bonds in frictionless markets, which operate in every period. The central issue is to compare equity and bond returns and risk premia under the two consumption scenarios along *rational expectations equilibria*. Since the consumer optimization problem cannot be solved analytically, we rely on numerical

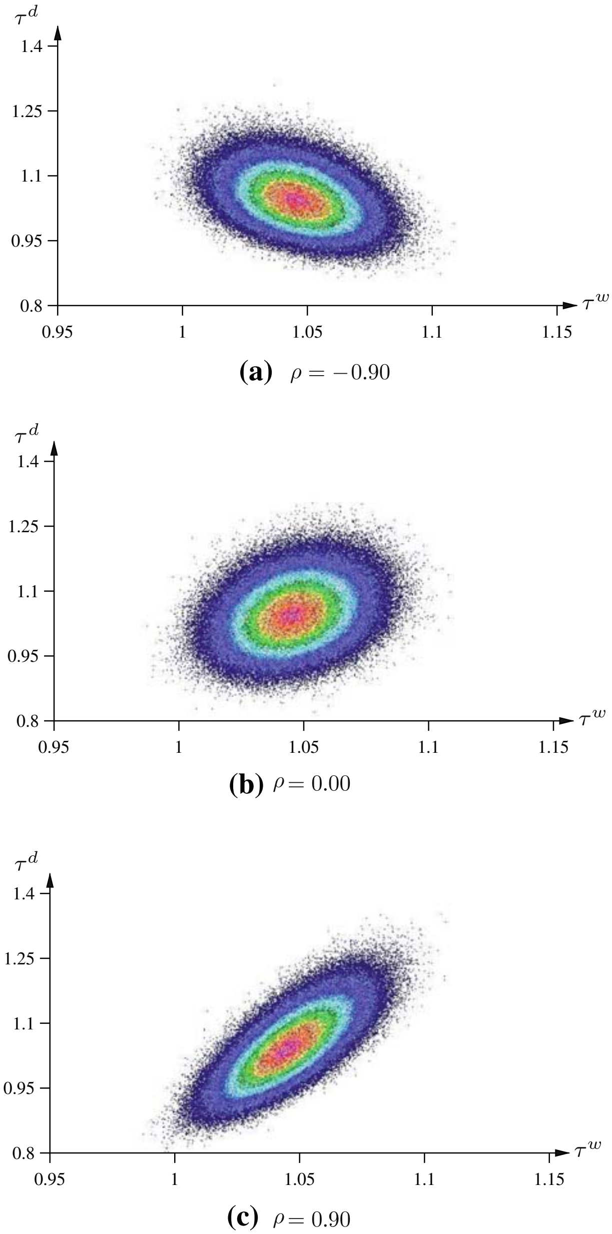


Fig. 2 Joint density of annualized growth rates of wages and dividends

procedures. We compute the processes of equity and bond returns induced by the sample path of the production sector. Given the stability of the random fixed point, it follows from ergodicity that the sample paths of asset returns converge as well to a stationary solution, whose statistical properties can be calculated again from the sample properties of the solution.

2.2.1 OLG Consumers

Consider first the standard OLG framework where each generation of consumers lives for two consecutive periods. They supply one unit of labor and receive wage income only in the first period, but consume in both periods. Purchasing x_t units of shares at price p_t in any period t implies a random cum-dividend return $x_t(p_{t+1} + d_{t+1})$ in the following period. Purchasing b_t units of the discount bond at price q_t implies one unit of return in the following period. Therefore, the consumption when young and old (c_t^y, c_{t+1}^o) are restricted by

$$c_t^y + x_t p_t + b_t q_t \leq w_t \quad \text{and} \quad c_{t+1}^o \leq x_t(p_{t+1} + d_{t+1}) + b_t. \tag{11}$$

Young consumers maximize

$$u(c_t^y) + \beta \mathbb{E} u(c_{t+1}^o), \tag{12}$$

subject to (11).

Joining the OLG consumption sector and the production sector of the previous section into a competitive system with market clearing, one obtains the description of an economy where the production sector is described by seven parameters $(\delta, \theta, \mu_a, \sigma_a, \mu_\eta, \sigma_\eta, \rho)$ and the consumption sector by two, (β, γ). For notational convenience it is useful to describe the state of the economy at any period t by the six dimensional vector $\omega_t = (A_t, k_t, a_t, \eta_t, \xi_t, g_t)$, whose evolution is governed by the system Eq. 5. Since the random system $(a_t, \eta_t, \xi_t, g_t)$ converges to a unique (asymptotically stable) stationary solution, it is natural to define the rational expectations equilibria as the associated minimum state variable (MSV) solution,[7] describing the supporting equity and bond prices under optimal behavior of OLG consumers.

Definition 1 A rational expectations equilibrium of the economy is a pair of price functions $(p(\omega), q(\omega))$ and a pair of consumption functions $(c^y(\omega), c^o(\omega))$ such that

- for the given pair of equity and bond price functions, $(p(\omega_t), q(\omega_t))$, young and old consumers' consumption plans satisfy the first order optimality conditions

$$p(\omega_t)u'(c^y(\omega_t)) = \beta \mathbb{E}\left[u'(c^o(\omega_{t+1}))(p(\omega_{t+1}) + d(\omega_{t+1}))\right] \tag{13}$$
$$q(\omega_t)u'(c^y(\omega_t)) = \beta \mathbb{E}\left[u'(c^o(\omega_{t+1}))\right]. \tag{14}$$

- for the given pair of consumption functions, $(c^y(\omega_t), c^o(\omega_t))$, equity and bond markets clear, i.e.

$$c^y(\omega_t) = w(\omega_t) - p(\omega_t) \quad \text{and} \quad c^o(\omega_t) = p(\omega_t) + d(\omega_t). \tag{15}$$

[7] This was first proposed in McCallum (1983) and also called the "functional rational expectations solution". See also Spear (1988), McCallum (1998, 1999), Böhm and Wenzelburger (2004) and Wenzelburger (2006).

Observe that the MSV solution to be derived builds on the feature of the stochastic properties of the production sector of the economy. The somewhat simplifying assumption made for the dividend payout ratio implies that investment decisions taken by the firm are independent of optimal consumption decisions. This allows a direct and unambiguous comparison between the returns on assets and risk premia induced by the two consumption scenarios.

Since equity shares are assumed to be in positive net supply (normalized to 1) while bonds are in zero net supply, market clearing portfolio holdings by young consumers are always $(x_t, b_t) = (1, 0)$ in each period and state ω_t. With these restrictions one obtains a substantial simplification of the system of functional equations required in Definition 1. Let $s(\omega_t)$ denote the equity price to wage ratio. Then, the equity price function has the multiplicative form $p(\omega_t) = w(\omega_t)s(\omega_t)$ and the consumption plans of young and old agents are $c^y(\omega_t) = w(\omega_t)(1-s(\omega_t))$ and $c^o(\omega_t) = p(\omega_t)+d(\omega_t) = w(\omega_t)(s(\omega_t) + \tau(\omega_t))$, where $\tau(\omega_t)$ is the dividend to wage ratio which is given by

$$\tau(\omega_t) := \frac{d(A,k)}{w(A,k)} = \eta(\omega_t)\frac{\pi(A_t,k_t)}{w(A_t,k_t)} = \eta(\omega_t)\frac{\theta}{1-\theta}. \tag{16}$$

Then, Assumption (10) on preferences implies that the Euler Eq. 13 can be rewritten as

$$\frac{p(\omega_t)}{(w(\omega_t) - p(\omega_t))^{\gamma}} = \beta \int_{\mathbb{R}^2} (p(\omega_{t+1}) + d(\omega_{t+1}))^{1-\gamma}\, dF(\omega_{t+1}|\omega_t) \tag{17}$$

where ω_{t+1} is next period's state and $F(\omega_{t+1}|\omega_t)$ is the one period state transition distribution function. To obtain the MSV solution, we divide both sides of Eq. 17 by $[w(\omega_t)]^{1-\gamma}$ and obtain

$$\frac{s(\omega_t)}{(1-s(\omega_t))^{\gamma}} = \beta \int_{\mathbb{R}^2} (s(\omega_{t+1})+\tau(\omega_{t+1}))^{1-\gamma} \left(\frac{w(\omega_{t+1})}{w(\omega_t)}\right)^{1-\gamma} dF(\omega_{t+1}|\omega_t). \tag{18}$$

Since the wage ratio is given by

$$\frac{w(\omega_{t+1})}{w(\omega_t)} = \frac{A_{t+1}}{A_t}\frac{k_{t+1}^{\theta}}{k_t^{\theta}} = e^{a_{t+1}} g_{t+1}^{\theta} = e^{a_{t+1}} ((1-\delta) + (1-\eta_t)\xi_t)^{\theta}, \tag{19}$$

the MSV solution of the integral Eq. 18 depends only on the pair (η_t, ξ_t). Analogously, Eqs. 14 and 15 imply that the discount bond price also depends only on the pair (η_t, ξ_t) and is given by

$$q(\omega_t) = \beta \int_{\mathbb{R}^2} \left(\frac{1-s(\omega_t)}{s(\omega_{t+1}) + \tau(\omega_{t+1})}\right)^{\gamma} \left(\frac{w(\omega_t)}{w(\omega_{t+1})}\right)^{\gamma} dF(\omega_{t+1}|\omega_t), \tag{20}$$

with the wage ratio as given in Eq. 19.

The above functional equations can be solved by relying on numerical procedures, which are quite involved, but nevertheless straightforward. Let $L(s)$ denote the following functional

$$L(s)(\omega_t) := \beta \int\limits_{\mathbb{R}^2} (s(\omega_{t+1}) + \tau(\omega_{t+1}))^{1-\gamma} \left(e^{a_{t+1}} g_{t+1}^{\theta}\right)^{1-\gamma} dF(\omega_{t+1}|\omega_t), \quad (21)$$

where a_{t+1}, η_{t+1}, and τ_{t+1} are defined in (5) and (16). In order to solve the functional Eq. 18, we write

$$f(s(\omega)) = L(s)(\omega), \quad \text{where } f(x) = \frac{x}{(1-x)^{\gamma}}. \quad (22)$$

We construct the state space, discretize it, and initialize $s_0(\omega) := 0$. At each collocation point we solve $f(s_n(\omega)) = L(s_{n-1})(\omega)$ for s_n. For the numerical interpolation of $s_n(\omega)$, we use Chebyshev polynomials of the first kind (see Judd (1998) for details). The numerical procedure generates the point wise solutions of the relevant state variables under the MSV solution with rational expectations as functions of the pair of random variables (η_t, ξ_t) which converges asymptotically to a unique random fixed point (stationary solution). Since the underlying stochastic process is unique and ergodic, its statistical properties are obtainable from the data generated by any chosen seed of random perturbations. Thus, the numerical investigation of the limiting properties of one orbit is sufficient to provide a complete description of the model.

2.2.2 Infinitely Lived Consumers

Consider an infinitely lived consumer who maximizes expected discounted utility

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}$$

where c_t is subject to the budget constraint $c_t + x_t p_t + b_t q_t \leq w_t + x_{t-1}(p_t + d_t) + b_{t-1}$, where x_t is the number of equity shares purchased in the equity market at price p_t, b_t is the purchase of units of discount bonds at price q_t, and (x_{t-1}, b_{t-1}) is the portfolio carried from the previous period. Thus, the consumption sector is represented again by two parameters (β, γ).

Given the structure of the economy, a rational expectations equilibrium is defined in the same way as in the OLG case as the MSV solution of the supporting price functions satisfying optimality and feasibility along an entire orbit.

Definition 2 A rational expectations equilibrium of the economy is a pair of price functions $(p(\omega), q(\omega))$ and a consumption allocation function $c(\omega)$ such that

- for the given consumption function $c(\omega)$, the pair of equity and bond price functions $(p(\omega), q(\omega))$ satisfy the optimality conditions

$$p(\omega_t)u'(c(\omega_t)) = \beta\mathbb{E}\left[u'(c(\omega_{t+1}))(p(\omega_{t+1}) + d(\omega_{t+1}))\right] \tag{23}$$

$$q(\omega_t)u'(c(\omega_t)) = \beta\mathbb{E}\left[u'(c(\omega_{t+1}))\right]. \tag{24}$$

- the consumption function $c(\omega_t)$ satisfies the feasibility constraint, i.e.

$$c(\omega_t) = w(\omega_t) + d(\omega_t). \tag{25}$$

Let $s(\omega_t)$ denote the equity price to wage ratio. Then the consumer's consumption plan is given by $c(\omega_t) = w(\omega_t) + d(\omega_t) = w(\omega_t)(1 + \tau(\omega_t))$, where $\tau(\omega_t)$ is defined by Eq. 16. Assumption (10) on preferences implies that the Euler Eq. 23 can be rewritten as

$$\frac{p(\omega_t)}{(w(\omega_t) + d(\omega_t))^\gamma} = \beta \int_{\mathbb{R}^2} \frac{p(\omega_{t+1}) + d(\omega_{t+1})}{(w(\omega_{t+1}) + d(\omega_{t+1}))^\gamma} dF(\omega_{t+1}|\omega_t), \tag{26}$$

where as above, ω_{t+1} denotes next period's state and $F(\omega_{t+1}|\omega_t)$ is the one period state transition distribution function. Dividing both sides of Eq. 26 by $[w(\omega_t)]^{1-\gamma}$ we obtain

$$s(\omega_t) = \beta \int_{\mathbb{R}^2} (s(\omega_{t+1}) + \tau(\omega_{t+1}))\left(\frac{1 + \tau(\omega_t)}{1 + \tau(\omega_{t+1})}\right)^\gamma \left(\frac{w(\omega_{t+1})}{w(\omega_t)}\right)^{1-\gamma} dF(\omega_{t+1}|\omega_t). \tag{27}$$

Since the wage ratio is given by Eq. 19, it follows that the MSV solution of the integral Eq. 27 depends only on the pair (η_t, ξ_t). Analogously, the bond price depends only on the pair (η_t, ξ_t) and is given by

$$q(\omega_t) = \beta \int_{\mathbb{R}^2} \left(\frac{w(\omega_t)}{w(\omega_{t+1})}\right)^\gamma \left(\frac{1 + \tau(\omega_t)}{1 + \tau(\omega_{t+1})}\right)^\gamma dF(\omega_{t+1}|\omega_t), \tag{28}$$

where the wage ratio is given in Eq. 19 and the dividend to wage ratio is defined in Eq. 16. Thus, one finds, as in the case with overlapping generations of consumers, that the consumption and pricing process with a representative consumer is given by a list of *deterministic* functions inducing random variables generated by the same productivity and growth process.

In order to solve the system of functional equations given in Eqs. 27 and 28 we rely again on numerical procedures. Now, let $L(s)$ denote the functional

$$L(s)(\omega_t)=\beta\int\limits_{\mathbb{R}^2}(s(\omega_{t+1})+\tau(\omega_{t+1}))\left(\frac{1+\tau(\omega_t)}{1+\tau(\omega_{t+1})}\right)^{\gamma}\left(\frac{w(\omega_{t+1})}{w(\omega_t)}\right)^{1-\gamma}dF(\omega_{t+1}|\omega_t) \tag{29}$$

where a_{t+1}, η_{t+1}, and τ_{t+1} are defined in (5) and (16). In order to solve the functional equation

$$s(\omega)=L(s)(\omega) \tag{30}$$

we proceed as before. We construct the state space, discretize it, and initialize $s_0(\omega):=0$. At each collocation point we solve, $s_n(\omega)=L(s_{n-1})(\omega)$, for s_n, For numerical interpolation of the $s_n(\omega)$ function, we use Chebyshev polynomials of the first kind again.

3 Comparing OLG and ILA Models

In order to compare the performance of the OLG model and the ILA model we solve Eqs. 18 and 20 and Eqs. 27 and 28 numerically. We use the standard parameter set and in addition we assume that consumer's annual time discount is 1/1.011 (see Hurd (1989) for empirical estimation) implying the discount factor for 35 years to be $(1/1.011)^{35}\approx 0.70$, and that the consumer's relative risk aversion γ is in the range of 1 to 4.[8] After solving Eqs. 18 and 27 numerically for $\gamma=1.00, 2.00, 3.00$, and 4.00, we generate the time series of equity and bond returns. Given the ergodicity of the system, we obtain unbiased estimates of all moments of the invariant distributions.[9] Table 4 summarizes the numerical findings for the first two moments of the risk premium implied by the two models indicating higher mean and lower standard deviations for the OLG model for all values of risk aversion. The risk premia reported in the table are in line with the findings of pp. 931–932 in Mehra and Prescott (2003), who report a risk premium of 0.35 for a reasonable parameter configuration. Table 5 displays the annualized percentage returns of equity and bonds for different levels of risk aversion. The returns implied by the OLG model are always lower than those implied by the ILA model. In addition, equity and bond returns implied by the OLG model display a low sensitivity with respect to risk aversion. Both models seem to explain well empirically observed equity returns for a risk aversion parameter close to unity. Both models fail, however, to predict the bond rate in a reasonable range (see Table 1 on pp. 891–892 in Mehra and Prescott 2003).

For degrees of risk aversion higher than one the implied mean returns and their volatility by the ILA model increase far beyond the values of the OLG model.

[8] See Arrow (1971), Mankiw (1981), Mehra and Prescott (1985), Skinner (1985), Attanasio and Weber (1989) and Zeldes (1989) for empirical evidence.

[9] The literature often uses local linearizations or approximations. For example, Kydland and Prescott (1982) use a linear-quadratic approximation of the Euler equation, Christiano (1990), Campbell (1994), and Lettau (2003) use a log-linear-quadratic and log-linear approximations, respectively. Due to the non linearity of the global mapping these linear approximations generate biased estimates of sample moments in general.

Table 4 Mean and SD of annualized percentage risk premium

	$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$R - r$: OLG	(0.01, 0.32)	(0.06, 0.53)	(0.14, 0.73)	(0.43, 0.92)
$R - r$: ILA	(0.00, 0.48)	(0.00, 1.05)	(0.02, 1.42)	(0.06, 1.79)

Table 5 Mean and SD of annualized percentage asset returns

	$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
R: OLG	(5.54, 0.29)	(6.11, 0.40)	(6.48, 0.51)	(6.76, 0.61)
R: ILA	(5.70, 0.37)	(10.56, 0.74)	(15.65, 0.94)	(20.97, 1.14)
r: OLG	(5.53, 0.13)	(6.06, 0.22)	(6.33, 0.31)	(6.34, 0.40)
r: ILA	(5.69, 0.19)	(10.56, 0.39)	(15.63, 0.62)	(20.91, 0.86)

A possible explanation for this effect can be found by comparing Eqs. 14 and 24. These imply that the bond rate is determined by the appropriate expectations of the two terms of the growth rates of consumption $(c_t/c_{t+1})^{\gamma}$ for the ILA model and by $(c_t^y/c_{t+1}^o)^{\gamma}$ for the OLG model. Numerically one finds that the second one is always more volatile. Therefore, with risk aversion $\gamma > 1$ the predicted bond rate becomes more sensitive with respect to the volatility of consumption growth, so that one observes a systematically higher bond rate in the ILA model. It is known also that under expected intertemporal utility maximization the supporting rates of return for equity and bonds always move together. This happens because in this framework one cannot separate the intertemporal substitution and the risk aversion effects. As a consequence we also observe the large deviation of equity returns for the ILA model from those in the OLG model.

In order to understand the effect of productivity growth and of the dividend payout ratio on equity and bond returns we distinguish between a direct and an indirect effect. Depending on the realization of productivity growth, the direct effect increases or decreases productivity of labor and capital and consequently affects wage and dividend payments for a given capital stock. The indirect effect operates through capital accumulation. In particular, high/low realization of the rate of productivity growth leads to a higher/lower accumulation of capital which has an effect on next period's wage and dividend incomes. Similar effects are created by the dividend payout ratio. In order to study when and in which model there is a stronger direct or indirect effect we have to rely on numerical simulations again. The same issue of measuring the direct and indirect effects on asset returns has been addressed in Lettau (2003) under a linear approximation.

Table 6 displays the direct effect of productivity and of the dividend payout ratio on equity returns. The correlation between productivity growth and the dividend payout ratio is high. However, it weakens as risk aversion increases. Table 7 displays the direct effect of productivity and the dividend payout ratio on the bond return. It shows high correlation between the bond rate and the realized dividend payout ratio, which makes the dividend payout ratio an almost perfect predictive device for the bond return.

In contrast to these findings, Tables 8 and 9 portray the indirect effect on equity and bond returns. Table 8 indicates that the correlation between equity return and

Table 6 Direct effect on equity returns: OLG versus ILA

		$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$corr(R, a)$	OLG model	0.72	0.47	0.34	0.27
	ILA model	0.57	0.29	0.24	0.20
$corr(R, \eta)$	OLG model	0.54	0.69	0.70	0.69
	ILA model	0.64	0.78	0.69	0.60

Table 7 Direct effect on bond returns: OLG versus ILA

		$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$corr(r, a)$	OLG model	0.54	0.45	0.40	0.36
	ILA model	0.36	0.36	0.36	0.36
$corr(r, \eta)$	OLG model	−0.67	−0.78	−0.83	−0.86
	ILA model	−0.86	−0.86	−0.86	−0.86

Table 8 Indirect effect on equity returns: OLG versus ILA

		$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$corr(R, g)$	OLG model	0.42	0.53	0.59	0.62
	ILA model	0.47	0.50	0.62	0.71
$corr(R, \xi)$	OLG model	0.46	0.20	0.06	−0.03
	ILA model	0.24	0.02	−0.05	−0.11

Table 9 Indirect effect on bond returns: OLG versus ILA

		$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$corr(r, g)$	OLG model	−0.11	−0.09	−0.08	−0.07
	ILA model	−0.07	−0.07	−0.07	−0.07
$corr(r, \xi)$	OLG model	0.74	0.63	0.55	0.50
	ILA model	0.50	0.50	0.50	0.50

capital growth is strong. It becomes stronger as risk aversion increases. However, the correlation between equity return and the rate of profit is weak and it becomes even weaker as the value of γ increases. Table 9 shows that the correlation between equity return and capital growth is very weak. In contrast, the correlation between the bond return and the rate of profit is strong.

The results above assume that there is no correlation between productivity and the dividend payout ratio, $\rho = 0$. This implies 27% correlation between wage and dividend growth rates and higher risk premia implied by the OLG model. Tables 10 and 11 display the annualized average risk premium and its volatility when $\rho = 0.20$ and $\rho = -0.20$. As the Tables indicate the main findings are robust with respect to this parameter variation. The risk premium prediction improves when $\rho = 0.20$ and worsens when $\rho = -0.20$. However, the implied differences are insignificant.

Table 10 Mean and SD of annualized percentage risk premium, $\rho = 0.20$

	$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$R - r$: OLG	(0.01, 0.34)	(0.07, 0.54)	(0.16, 0.73)	(0.49, 0.91)
$R - r$: ILA	(0.01, 0.50)	(0.01, 1.05)	(0.04, 1.41)	(0.09, 1.76)

Table 11 Mean and SD of annualized percentage risk premium, $\rho = -0.20$

	$\gamma = 1.00$	$\gamma = 2.00$	$\gamma = 3.00$	$\gamma = 4.00$
$R - r$: OLG	(0.01, 0.30)	(0.05, 0.52)	(0.12, 0.73)	(0.37, 0.92)
$R - r$: ILA	(0.00, 0.46)	(0.00, 1.04)	(0.00, 1.43)	(0.03, 1.82)

4 Summary and Conclusions

The comparison of the two leading macro economic models of asset pricing was carried out for a general stochastic growth model with a random productivity growth process. Since the dividend payout ratio is exogenous and stochastic, the firm's investment decision becomes independent of consumer savings decisions. As a result a direct and unambiguous comparison between the returns on assets and risk premia induced by the two consumption scenarios is possible. Within this scenario it has been shown that the risk premium predicted by the OLG model is slightly better than the one by the ILA model relative to empirically observed average values. However, this difference becomes insignificant statistically as soon as the volatility of the risk premium is taken into consideration. The equity returns in both scenarios are almost the same for $\gamma = 1$. However, as γ increases, the values predicted by the OLG become compatible with empirically observed averages, while the ILA model overpredicts systematically. In contrast, the bond returns predicted in both scenarios are too high compared with empirical data.

Our numerical experiments show that the results are robust with respect to changes of the correlation parameter ρ. However, the results depend crucially on the assumption that the dividend payout ratio is truly stochastic. When the dividend payout ratio is constant some of the above results are reversed and the ILA model performs better than the OLG model on several counts. In particular, risk premia implied by the ILA model are higher than the ones of the OLG model (see Böhm et al. 2006a). A possible explanation for this effect is that a constant dividend payout ratio implies a perfect correlation between the growth rates of wages and dividends. Therefore, OLG consumers require lower risk premia for equity because they don't face wage income risk when old.

Concerning equity returns, the OLG model predicts empirically observed averages quite well for reasonable parameter values, while the ILA model overpredicts equity returns as soon as $\gamma \geq 2$. The bond rates are overpredicted by both models, especially by the ILA model. Asset returns implied by the ILA model display a stronger positive relation with risk aversion parameter, while this relation is very small in the OLG model. In both models equity return data are highly correlated with the growth rate of productivity. This correlation weakens for high values of relative risk aversion. As the relative risk aversion increases the correlation between capital growth and equity return becomes strong, making the realized capital growth an almost perfect predictor for equity market performance. In contrast, the realization of the dividend payout ratio can be viewed as a unique and almost perfect predictor for the bond return. The results obtained in the paper are robust in particular with respect to changes of ρ, the correlation of the shocks of productivity growth and the dividend pay out ratio.

It is left for future research to investigate to what extent the feed back from preferences and risk aversion on real investment is a determining factor for asset returns. It is known from deterministic models[10] that such interaction can be crucial for the long run behavior of growth and asset returns. Such an extension would allow one to analyze both, firms and consumers within an optimizing intertemporal behavior. Moreover, one could obtain a more complete characterization of the links or interactions between real and financial markets.

Acknowledgements This paper was written as part of the project "International Financial Markets and Economic Development of Nations" supported by the Deutsche Forschungsgemeinschaft under contract Bo. 635/12-1. We thank the Guest Editor Silvano Cincotti and four anonymous referees for very helpful comments and suggestions.

References

Arnold, L.(1998). *Random dynamical systems*. Berlin: Springer-Verlag.

Arrow, K. (1971). *Essays in the theory of risk bearing*. North-Holland, Amsterdam.

Attanasio, O. P., & Weber, G. (1989). Intertemporal substitution, risk aversion and euler equation for consumption. *The Economic Journal, 99*, 57–73.

Barro, R. (1974). Are government bonds net wealth? *Journal of Political Economy, 82*, 1095–1117.

Blanchard, O. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy, 93*, 223–247.

Blanchard, O. J., & Fischer, S. (1989). *Lectures on macroeconomics*. Cambridge: MIT Press.

Böhm, V. (2003). MACRODYN*—the handbook. Discussion paper no. 498*. Bielefeld: Department of Economics, Bielefeld University.

Böhm, V., Kikuchi, T., & Vachadze, G. (2005). *Welfare and the role of equity in an economy with capital accumulation. Discussion paper 547*. Department of Economics, Bielefeld University.

Böhm, V., Kikuchi, T., & Vachadze, G. (2006a). *Asset pricing with Markovian productivity growth. Discussion paper 558*. Department of Economics, Bielefeld University.

Böhm, V., Kikuchi, T., & Vachadze, G. (2006b). *On the role of equity for the dynamics of capital accumulation. Discussion paper 551*. Department of Economics, Bielefeld University.

Böhm, V., & Wenzelburger, J. (2002). Perfect predictions in economic dynamical systems with random perturbations. *Macroeconomic Dynamics, 6*(5), 687–712.

Böhm, V., & Wenzelburger, J. (2004). Expectational leads in economic dynamical systems. In W. Barnett, C. Deissenberg, & G. Feichtinger (Eds.), *Economic complexity: Non-linear dynamics, multi-agents economies, and learning* (pp. 333–361). Amsterdam: Elsevier.

Campbell, J. Y. (1994). Inspecting the mechanism: An analytical approach to the stochastic growth model. *Journal of Monetary Economics, 33*(3), 463–506.

Christiano, L. (1990). Linear-quadratic approximation and value-function iteration: A comparison. *Journal of Business Economics and Statistics, 8*, 99–113.

Constantinides, G. M., Donaldson, J. B., & Mehra, R. (2002). Junior can't borrow: A new perspective on the equity premium puzzle. *Quarterly Journal of Economics, 117*, 269–296.

Hillebrand, M., & Wenzelburger, J. (2006). On the dynamics of asset prices and portfolios in a multiperiod CAPM. *Chaos, Solitons and Fractals, 29(3)*, 578–594.

[10] See for example Böhm et al. (2005, 2006a).

Howarth, R. (1996). Climate change and overlapping generations. *Contemporary Economic Policy, 14*, 100–111.
Howarth, R. (1998). An overlapping generations model of climate–economy interactions. *Scandinavian Journal of Economics, 100*, 575–591.
Huggett, M. (1996). Wealth distribution in life-cycle economies. *Journal of Monetary Economics, 38*, 469–494.
Hurd, M. (1989). Mortality risk and bequests. *Econometrica, 57*, 173–209.
Judd, K. L. (1998). *Numerical methods in economics*. Cambridge: MIT Press.
Kydland, F. E., & Prescott, E. C. (1982). Time to build and aggregate fluctation. *Econometrica, 50*(6), 1345–1370.
Lettau, M. (2003). Inspecting the mechanism: Closed-form solutions for asset prices in real business cycle models. *The Economic Journal, 113*, 550–575.
Lintner, J. M. (1956). Distribution of incomes of corporations among dividends, retained earnings and taxes. *American Economic Review, 46*, 97–113.
Mankiw, N. (1981). The permanent income hypothesis and the real interest rate. *Economics Letters, 7*, 307–311.
McCallum, B. T. (1983). On non-uniqueness in rational expectations models—an attempt at perspective. *Journal of Monetary Economics, 11*, 139–168.
McCallum, B. T. (1998). Solutions to linear rational expectations models: A compact exposition. *Economics Letters, 61*, 143–147.
McCallum, B. T. (1999). Role of the minimal state variable criterion in rational expectations models. *International Tax and Public Finance, 6*, 621–639.
Mehra, R., & Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics, 15*(2), 145–161.
Mehra, R., & Prescott, E. C. (2003). The equity premium in retrospect. In G. Constantinides, M. Harris, & R. Stulz (Eds.), *Handbook of the economics of finance*. North-Holland, Amsterdam, Netherlands.
Müller-Fürstenberger, S. G., & Predivoli, P. (1997). Overlapping generations or infinitely lived agents. *Environmental and Resource Economics, 10*, 27–40.
Prescott, E. C. (1986). Theory ahead of business cycle measurement. *Federal Reserve Bank of Minneapolis Quarterly Review, 10*(4), 9–22.
Ríos, J. V. (1994). On the quantitative importance of market completeness. *Journal of Monetary Economics, 34*, 463–496.
Ríos, J. V. (1996). Life-cycle economies and aggregate fluctuations. *The Review of Economic Studies, 63*, 465–489.
Schmalfuss, B. (1996). A random fixed point theorem based on lyapunov exponents. *Random and Computational Dynamics, 4*(4), 257–268.
Schmalfuss, B. (1998). A random fixed point theorem and the random graph transformation. *Journal of Mathematical Analysis and Application, 225*(1), 91–113.
Skinner, J. (1985). Variable lifespan and the intertemporal elasticity of consumption. *Review of Economics and Statistics, 67*, 616–623.
Spear, S. (1988). Existence and local uniqueness of functional rational expectations equilibria in dynamic economic system models. *Journal of Economic Theory, 44*(1), 124–155.
Storesletten, K., Telmer, C. I., & Yaron, A. (2007). Asset pricing with idiosyncratic risk and overlapping generations. *Review of Economic Dynamics, 10*(4), 519–548.
Wenzelburger, J. (2006). *Learning in economic systems with expectations feedback* (Vol. 555). Springer.
Zeldes, S. P. (1989). Optimal consumption with stochastic income. *Quarterly Journal of Economics, 104*, 275–298.