Inequality of Nations and Endogenous Fluctuations in a Two Country Model*

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February 5, 2008

Abstract

This paper examines how international financial interaction influences the global dynamics of the Matsuyama model (Econometrica 72, 2004) under the alternative assumption that the world consists of two economies, instead of a continuum of small open economies. The spillover effects of these two economies, which each have a different population size, generate a stable steady state. In this steady state the world is divided into a rich economy with a small population and a poor economy with a large population. This steady state undergoes a bifurcation inducing fluctuations of financial flows between the poor economy and the rich economy endogenously.

Keywords: Credit market imperfection, Endogenous cycles, Symmetry-breaking, Two-country model

JEL classification: E44, F43, O11

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*I acknowledge the financial support of the German Research Foundation (DFG) Bo. 635/12–1. Earlier versions of the paper were presented in seminars at Bielefeld, Tel Aviv, Bonn, St Andrews and NUS and the QED Workshop in Bielefeld, 2005, the Meeting on Macroeconomics and Development in Aix en Provence, 2006, the Far Eastern Meeting of the Econometric Society in Beijing, 2006, the European Meeting of the Econometric Society in Vienna, 2006, and the Annual Congress of the Verein für Socialpolitik, Bayreut, 2006. In particular, I wish to thank Volker Böhm, Thorsten Pampel, George Vachadze, Jan Wenzelburger and Itzhak Zilcha for helpful discussions and comments. I am indebted to Kiminori Matsuyama and an anonymous referee for many useful and constructive suggestions which improved the paper significantly.

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1 Introduction

The richest 25 percent of the world’s population receives 75 percent of the world’s income even when adjusting for purchasing power parity (see Milanovic 2002). Logically it follows that the poorest 75 percent of the world’s population shares just 25 percent of the world’s income. There must be a connection between these two groups of people, and the groups of countries in which they live. In fact, the interaction between these two groups might explain a wide range of economic phenomena. The present paper shows that fluctuations of income arise endogenously from the interaction in the international financial market between two groups: a poor group with a large population and a rich group with a small population. These two groups differ only in their initial levels of capital stock and in their size of population. In the absence of the international financial market, they converge to the same level of capital stock. Therefore, it is the interaction of the two groups in the international financial market that is responsible not only for the inequality of the income but also for the fluctuation (volatility) of the income.

One common way to explain the cross country income difference is to use models with multiple equilibria. These models are sometimes referred to as “poverty trap” models, since the economy, caught in a vicious cycle, suffers from persistent underdevelopment. In a formal sense, these models have a threshold of initial conditions, above which the dynamical system converges to a high steady state, and below which to a low steady state. Since economies with identical structural characteristics may converge to different income levels, these models can be viewed as an alternative to models that attribute the cross country income difference to the cross country difference in structural characteristics. However, “poverty trap” models are closed economy models, and therefore it is not satisfactory as an explanation for the cross country income difference if we wish to understand how the interaction between countries affects their income difference.

In an open economy setup, the neoclassical framework predicts that capital will flow from rich countries where the marginal product of capital is low, to poor countries where the marginal product of capital is high until the capital stocks in all countries

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1See Matsuyama (2005) for a recent survey on poverty trap models.
2In addition to identical structural characteristics, convergence to the same steady state requires similarity in initial conditions in these models. Hence, these models predict club convergence. See Galor (1996) for the debate about inferences from theoretical models on convergence issues.
3The Solow model predicts conditional convergence because countries converge to the same level of income if they have identical structural characteristics.
are equalized. Lucas (1990) was the first to suggest how to reconcile this neoclassical implication with empirical observations. Boyd & Smith (1997) and Matsuyama (2004) followed by building dynamic general equilibrium models which are capable of showing the capital flow from rich to poor countries. They extended the standard overlapping generations (OLG) model to incorporate imperfections in financial markets. In both models, the world economy consists of inherently identical countries, which differ only in their initial levels of capital. It is the wealth dependent borrowing constraint in Matsuyama (2004), and the external financing associated with a costly state verification problem in Boyd & Smith (1997), that counteracts the equalizing force of the diminishing marginal productivity. Both models show that symmetry breaking occurs in the presence of the international financial market. That is, the symmetric steady state loses its stability and stable asymmetric steady states come to exist.

This present paper extends Matsuyama (2004)'s model, in which the world consists of a continuum of small open economies, to a two country model. The main result of the letter can be characterized by two features: the spillover effect and the difference in population size of the two economies/countries. The spillover effect, which is the effect of a change in the capital stock of one economy on the capital stock of the other, is absent in Matsuyama's model since each small open economy has a zero measure. The two economies in the present model may be of a different population size while Boyd & Smith (1997) only analyze two countries with an identical population size. The spillover effect and the difference in population size affect the economies through the world interest rate which is determined endogenously by the demand and supply of the two economies. The present paper shows that this spillover effect and the difference in relative population size of the two economies generate a new stable steady state, in which the world is divided into a rich economy with a small population and a poor economy with a large population. Furthermore, this steady state induces the fluctuation of income and international financial flows endogenously.

While the widely accepted real business cycles (RBC) models understand fluctuations as an adjustment process to a steady state, the two country model described here reveals how the interaction in the international financial market causes the income to fluctuate in the long run. This outcome is in contrast to the result obtained by Boyd & Smith (1997) who motivate their paper by referring to the cyclicality of credit allocation between developing and developed economies in empirical data. Their theoretical finding is a dynamical equilibrium path displaying damped oscillation which eventually converges
to a steady state.

Matsuyama argues in his paper that the symmetry breaking approach offers different policy implications from the poverty trap approach since the case of underdevelopment is no longer an isolated problem, which can be treated independently for each country, but a part of the interrelated whole. Similarly, the endogenous cycles approach offers different policy implications from the RBC approach since fluctuation is not caused by a propagation of random shocks, which may be controlled independently by each country, but generated by the interaction of economies. Therefore, the two country model provides a unified framework, within which not only the inequality of nations, but also the fluctuation of the income can be accounted for, and explained, endogenously. In addition, it is important to highlight the basic conceptual difference between poverty and inequality. Closed economy models may provide policy implications for the reduction of poverty but not for the reduction of inequality. The question is why inequality arises even when the two countries are fully integrated through a competitive financial market. The two country model presented here not only predicts that the inequality of income is a part of the interrelated whole, but also related to the fluctuation of income. Therefore, the inequality and fluctuation of income needs to be dealt with at the global as well as the integrated level.

This view that fluctuations are generated by the interaction of economies instead of a propagation mechanism of exogenous shocks is argued by Kikuchi (2007). However, there are structural differences between the two country model in Kikuchi (2007) and in the present paper. Kikuchi (2007)’s model includes an asset market, which diverts the savings away from productive investment. Nevertheless, the agents in the previous model hold an optimal portfolio inducing the rich country to provide credits to the poor country. This relationship is reversed in the present model where the borrowing constraint limits the domestic investment in the poor country forcing it to become a supplier of credit despite its higher marginal productivity.

The remainder of the paper is organized as follows. Section 2 shortly reviews the Matsuyama model. Section 3 generalizes the model to a two country case. Section 4 characterizes the stability property of all steady states of the two country model when the two countries have an identical population size. Then, Section 5 analyzes how the spillover effect and the difference in population size generate a new stable asymmetric steady state, and induce the fluctuation of income and international financial flows endogenously. Section 6 concludes.
2 The Matsuyama Model

This section reviews the model presented by Matsuyama (2004) and summarizes its findings. The Matsuyama model is the standard OLG model, modified only to incorporate credit market imperfection. There is a continuum of agents with unit mass. Each agent, who lives for two periods, supplies one unit of labor in the first period and consumes only in the second period. In order to transfer his income to the next period, the agent becomes either a borrower (investor) or a lender in the credit market.

**Assumption 1.** The production function in intensive form $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $C^2$, and satisfies $f(0) = 0$, $f''(k) < 0 < f'(k)$, and the Inada conditions $\lim_{k \to \infty} f'(k) = 0$ and $\lim_{k \to 0} f'(k) = \infty$.

Let the wage function be defined by $W(k) \equiv f(k) - kf'(k)$. To avoid multiple steady states, that are not related to credit market imperfection, the following assumption is imposed.

**Assumption 2.** $\lim_{k \to 0} W'(k) = \infty$ and $W''(k) < 0$.

It is assumed that to start up an investment project one unit of the consumption good is required. The investor is endowed with a linear technology which transforms one unit of the consumption good into $R$ units of physical capital. The assumption $W(R) < 1$ makes sure that investors always have to borrow $1 - W(k)$ in the credit market. There exist two constraints which must always hold in the credit market. Firstly, there exists a borrowing constraint such that the investor cannot borrow more than a fraction $\lambda \in (0, 1)$ of the revenue of the investment project. Secondly, to start up a project, the return of investment has to be at least equal to the return from savings.

In autarky, the interest rate is adjusted so that the domestic saving is equal to the domestic investment. In other words, capital formation is not affected by the two constraints. Therefore, the dynamics of capital formation in autarky is fully described by

$$k_{t+1} = RW(k_t).$$

Assumption 2 ensures that Eq. (1) has a unique steady state $k = K^*(R) \in (0, R)$ defined implicitly by $k = RW(k)$, and $k_t$ converges monotonically to $k = K^*(R)$ for $k_0 \in \mathbb{R}_{++}$. The function $K^*(R)$ is increasing and satisfies $K^*(0) = 0$ and $K^*(R^+) = R^+$ where $R^+$ is defined by $W(R^+) = 1$. 


The world interest rate $r$ is given exogenously in the small open economy. This implies that the interest rate does not adjust to equate domestic saving and domestic investment. In other words, given the world interest rate, international borrowing and lending are adjusted so that the two constraints determine capital formation. Then, the dynamics of capital formation of the small open economy is fully described by

$$k_{t+1} = \Psi(k_t, r) \equiv \begin{cases} 
\Phi \left( \frac{r(1-W(k_t))}{\lambda R} \right) & \text{if } k_t < K(\lambda) \\
\Phi \left( \frac{r}{R} \right) & \text{if } k_t \geq K(\lambda)
\end{cases}$$

(2)

where $\Phi \equiv (f')^{-1}$, $\lambda \in (0, 1)$ is interpreted as a measure of imperfection in the financial market and $K(\lambda)$ is defined implicitly by $W(K(\lambda)) = 1 - \lambda$. For $k_t < K(\lambda)$, the borrowing constraint is binding in the credit market. Figure 1 depicts the time one map of the small open economy under certain parameter conditions showing a possibility of multiple steady states.

![Figure 1: Time one map of the small open economy](image)

The world economy consists of a continuum of small open economies with unit mass. Matsuyama argues that “in any steady state of the world economy each country must be at a stable steady state of the small open economy”. Eventually, what he shows is the existence of asymmetric steady states in which the world economy is divided into a fraction of economies with a low income $k_L$ and the rest with a high income $k_H$. His scenario of the symmetry breaking is the following. In the absence of the international financial market, each country is in autarky and converges to the same steady state. In
the presence of the international financial market, however, the symmetric steady state is unstable under certain parameter conditions. Suppose now that the economies in the world are hit by exogenous shocks. Since each economy is a small open economy and has no influence on the world interest rate alone, the world will be polarized into rich economies, which are hit by better shocks, and poor economies, which are hit by worse shocks. Figure 2 illustrates this scenario of symmetry breaking where the dotted line describes the evolution of the distribution of the capital stocks across economies and $x$ denotes the fraction of economies which have $k_L$ in the asymmetric steady state.

3 Two Country Model

Suppose now that the world consists of two “large” economies with identical structural characteristics. For the two country model we employ a Cobb-Douglas production function of the form $f(k) \equiv Ak^\alpha$ with $\alpha \in (0, 1)$ throughout the paper. Notice that the Cobb-Douglas production function satisfies Assumption 1 and 2. From Eq. (2) the capital investment in country $i$ is described by

$$k_{i,t+1} = \Psi(k_{i,t}, r_{t+1}), \ i = 1, 2.$$  

Equating total credit demand and total credit supply, the equilibrium interest rate $r_{t+1} = \mathcal{R}(k_{1,t}, k_{2,t})$ in the international financial market is implicitly defined by a solution
\[ L \Psi(k^1_t, r_{t+1}) + (1 - L) \Psi(k^2_t, r_{t+1}) = R(LW(k^1_t) + (1 - L)W(k^2_t)) \]  
\[ \text{where } L \in (0, 1) \text{ is defined to be the relative population size of country 1.} \]

For any \( k^1, k^2 > 0 \) the right hand side of Eq. (4) is a positive constant. The left hand side is monotonically decreasing in \( r \) since \( \Psi(k, r) \) is monotonically decreasing in \( r \). Since \( \lim_{r \to 0} \Psi(k, r) = \infty \) and \( \lim_{r \to \infty} \Psi(k, r) = 0 \), there exists a unique solution \( r = R(k^1, k^2) \).

Substituting the solution \( r_{t+1} = R(k^1_t, k^2_t) \) into Eq. (3) we can solve the two dimensional dynamical system:

\[ k^i_{t+1} = \Psi(k^i_t, R(k^1_t, k^2_t)), \quad i = 1, 2. \]  

In general, the two countries have a positive measure and the spillover effect

\[ \frac{\partial}{\partial k^i_t} R(k^1_t, k^2_t), \quad i = 1, 2 \]

is non-zero.\(^4\) Section 4 shows that this spillover effect generates new stable steady states in contrast to the Matsuyama model and Section 5 shows that the spillover effect generates endogenous cycles if we vary the relative population size of the two countries.

From Eqs. (3) and (4), the steady state of the two country model can be written as a pair \((k^1, k^2)\) which satisfies

\[ G(k^1, k^2) \equiv L(k^1 - RW(k^1)) + (1 - L)(k^2 - RW(k^2)) = 0 \]  
\[ \text{and} \]

\[ H(k^1) = H(k^2) \text{ if } k^1, k^2 < K(\lambda) \]  
\[ H(k^1) = f'(k^2) \text{ if } k^1 < K(\lambda) \leq k^2 \]  
\[ H(k^2) = f'(k^1) \text{ if } k^2 < K(\lambda) \leq k^1 \]  
\[ f'(k^1) = f'(k^2) \text{ if } k^1, k^2 \geq K(\lambda) \]

\(^4\)In the Matsuyama model, there is a continuum of homogeneous small open economies, hence the world interest rate is determined by the condition

\[ \int_0^1 \Psi(k^i_t, r_{t+1}) \, di = R \int_0^1 W(k^i_t) \, di. \]

This equation with Eq. (3) defines the dynamical system which is infinite-dimensional.

\(^5\)For the Cobb-Douglas production function we can solve for the world interest rate explicitly and thus we obtain the dynamical system explicitly.

\(^6\)The spillover effect is zero when there is a continuum of small open economies as no country has a positive measure.
where $H(k) \equiv \frac{\lambda f'(k)}{1 - W(k)}$. If $k^1, k^2 \geq K(\lambda)$, the steady state is unique and symmetric. Setting $L = 1$ in Eq. (4) induces the same steady state interest rate as setting $k^1 = k^2$. This means that the world interest rate in the symmetric steady state is identical to that of autarky. In particular, there exists a unique positive symmetric steady state $(K^*(R), K^*(R))$, where $K^*(R)$ coincides with the steady state of autarky. This implies that there are no financial transactions between the two economies in the symmetric steady state.

4 Two Countries with an Identical Population Size

In this section we analyze the existence and stability of all steady states when $L = 1/2$. In particular, the analysis shows that the spillover effect generates new asymmetric steady states in contrast to the Matsuyama model but does not generate endogenous cycles when $L = 1/2$.

**Proposition 1.** If $\lambda \geq \alpha$, there exists no asymmetric steady state.

See the appendix for a proof. For asymmetric steady states to exist, Proposition 1 says that the borrowing constraint of investors must be severe enough compared to the elasticity of production.

**Proposition 2.** Let $f(K^*(R_c)) \equiv 1$, $\phi(k) \equiv (f')^{-1}(H(k))$ and $R_{cc}$ be the value of $R$ for which $G(k^1, k^2) = 0$ is tangent to the graph of $\phi(k^i)$ for $i = 1, 2$. Suppose that $\lambda < \alpha$. There exist

1. two asymmetric steady states if and only if $R_c < R < (K^*)^{-1}(K(\lambda))$.

2. four asymmetric steady states if and only if $(K^*)^{-1}(K(\lambda)) < R < R_{cc}$.

See the appendix for a proof.

**Proposition 3.** The symmetric steady state $(K^*(R), K^*(R))$ is

1. locally stable if and only if $R < R_c$ or $R > (K^*)^{-1}(K(\lambda))$.

2. a saddle point if $R_c < R < (K^*)^{-1}(K(\lambda))$. 


See the appendix for a proof. Propositions 2 and 3 imply that the existence of two asymmetric steady states coincides with the instability of the symmetric steady state. Notice also that the existence of four asymmetric steady states coincides with the local stability of the symmetric steady state.

**Proposition 4.** If \( k_1^0 = k_2^0 > 0 \) or \( k_1^0, k_2^0 \geq K(\lambda) \), the system converges to the symmetric steady state \((K^*(R), K^*(R))\).

See the appendix for a proof. Proposition 4 says that if the initial capital stocks of the two economies are high enough so that they do not face the borrowing constraint, they will converge to the symmetric steady state. This will happen even if the symmetric steady state is unstable. This is because if \( k_1^0, k_2^0 \geq K(\lambda) \), the capital stocks in both economies adjust to the same level in the following period. Once the capital stocks in both economies are the same, there are no transactions between them and both follow a convergence path of the autarky economy.

Let us fully characterize the stability of the steady states by means of numerical simulation. The standard parameter set is given in Table 1 and will be used unless otherwise indicated. Figure 3 depicts the zero contours of the functions \( \Delta k_1(k_1, k_2) \equiv k_1 - \Psi_1(k_1, k_2) \) and \( \Delta k_2(k_1, k_2) \equiv k_2 - \Psi_2(k_1, k_2) \). The intersections of these two contours are the steady states of the model. For this parameter set, \( R_c = 2 \) and \((K^*)^{-1}(K(\lambda)) = 3.4\). For \( 0 < R \leq 2 \), the steady state is unique and symmetric as in (a). In addition to the symmetric steady state, there exist two asymmetric steady states for \( 2 < R \leq 3.4 \) as in (b) and (c), and four asymmetric steady states for \( R > 3.4 \) as in (d). The system is symmetric for \( L = 0.5 \) and therefore asymmetric steady states appear pairwise along the diagonal.

To analyze the stability of the steady states globally, basins of attraction are calculated and shown by different colors in Figure 4 for different values of \( R \). Figure 4 (a) indicates that the uniqueness of the symmetric steady state and its local stability (Propositions 2 and 3) together imply global stability. Figures 4 (b) and (c) indicate that the instability of the symmetric steady state implies stability of the two asymmetric steady states.
(Propositions 2 and 3). Finally, Figure 4 (d) indicates that the stability of the two asymmetric steady states and the symmetric steady state implies instability of the two additional asymmetric steady states (Compare Figure 3 (d) and Figure 4 (d)).

Let us now focus on the main results of this section. Figures 3 (a) and (b) are reproduced in Figure 5 as phase diagrams. Inside the dotted lines, \((k_1, k_2) < (K(\lambda), K(\lambda))\) so that the borrowing constraint is binding in both economies. While the symmetric steady state is globally stable in (a), it is a saddle point in (b) (Proposition 3). From Proposition 4 we know that if \(k_1^L = k_2^L > 0\) or \(k_1^M, k_2^M \geq K(\lambda)\) the world economy converges to this saddle point in (b) along the saddle path. The asymmetric steady states are locally stable. This is a typical case of symmetry breaking but this symmetry breaking can not be observed in Matsuyama (2004). Eq. (3) implies that at any steady state of the world economy each economy must be at a steady state of the small open economy. Suppose that the two economies were two small open economies. Then, one economy would have \(k_L\) and the other would have \(k_M\) in these asymmetric steady states since they lie within the dotted lines. Now from Figure 1 we know that \(k_L\) is stable but \(k_M\) is unstable. Therefore, the asymmetric steady state \((k_L, k_M)\) cannot be stable if the two economies
were small open economies. The difference lies in the assumption that in the present model the two economies are two “large” economies and have a positive measure. This implies that each economy influences not only its own capital stock but also the other’s through the world interest rate. In other words, the spillover effect, which is absent in the Matsuyama model, causes the symmetry breaking in this case.

5 Population Size and Endogenous Fluctuations

The stylized fact is that a small fraction of the world’s population occupies a large fraction of the world’s income while a large fraction of the world’s population occupies a small fraction of the world’s income. Firstly, the analysis in this section shows that the spillover effect generates a stable steady state in which the world is divided into a poor economy with a large population and a rich economy with a small population matching the stylized fact. Secondly, the numerical simulation of the model shows that this stable steady state undergoes a bifurcation generating endogenous cycles for a sufficiently unequal population size of the two economies. In other words, the spillover effect and the unequal population size together relate the inequality of income to the fluctuation of income and international financial flows.

Proposition 5. Let $L_c$ be the value of $L$ for which $H(k^1) = H(k^2)$ is tangent to $G(k^1, k^2) = 0$. Without loss of generality, let $L_c < 1/2$. If $L < L_c$, $R < R_c$ and $\lambda < \alpha$, there exist two asymmetric steady states, in which the economy with a smaller population is richer than the economy with a larger population.
See the appendix for a proof. If the two economies have an identical population size, Proposition 2 shows that there exists a unique symmetric steady state for $R < R_c$. In contrast, Proposition 5 says that if we change the relative population size of the two economies, two asymmetric steady states emerge even for $R < R_c$. Figure 6 (a) shows a phase diagram of the two country model with an unequal population size. The arrows in the figure show that the asymmetric steady state, which is more remote from the 45° line, is stable. This asymmetric steady state lies within the dotted lines indicating that in these steady states the borrowing constraint is binding and that the spillover effect must be at work to stabilize the steady state. This can be confirmed by looking at Figure 6 (b) which shows the basins of attraction for the respective steady states.

Let us investigate the effect of a change in the relative population size on the long run behavior of the system more globally. Figure 7 shows a bifurcation diagram for $k^1_t$ and $k^2_t$ with respect to $L$ depicting the limiting behavior of the state variables. It shows that greater difference in population size is associated with greater income inequality. The basins of attraction for the asymmetric steady state (the stable one in Proposition 5) expands, as the value of $L$ decreases, and includes the initial capital stock of the two countries at $L_a$. Then, the stable asymmetric steady state undergoes a bifurcation at $L_b$. There is a second bifurcation point at $L_{bb}$ at which $k^1 = K(\lambda)$. Let us first focus on the bifurcation point at $L_b$ where the borrowing constraint is binding in both economies.

**Proposition 6.** The stable asymmetric steady state in Proposition 5 undergoes a supercritical Neimark-Sacker bifurcation at $L \in (0, L_c)$.

See the appendix for a proof. From the previous discussion in Section 4 we know that
the spillover effect in the two country model is essential for the stability of the steady states when the borrowing constraint is binding in both economies. In addition, we know from Proposition 5 that it must be because of the difference in population size that the asymmetric steady states arise for $R < R_c$. This means that the difference in population size and the spillover effect jointly cause the bifurcation we observe. These two aspects of the two “large” economies absent in the world with a continuum of small open economies, generate the fluctuation of income and international financial flows endogenously.

Let us make a tentative explanation of the inherent structure of the model which induces endogenous cycles. Just as in a predator prey model think of two economies as two rivals, which are struggling for survival. There is a poor and a rich economy. Let us divide the state space into four areas. Figure 8 shows the lower right quadrant. This quadrant is again divided into four areas by lines $A$ and $B$.

The poor economy sustains a constant capital stock in autarky. However, in the presence of the international credit market, capital flows from the poor to the rich economy. In other words, the impoverishment of the poor economy is dependent on the rich economy. The richer is the rich economy, the stronger is its negative spillover effect. Figure 8 shows that this negative spillover effect prevails and impoverishes the poor if the rich economy’s capital stock is above $A$ while the poor economy grows below $A$. 

Figure 7: Bifurcation diagram $R = 1.8$. 

Let us divide the state space into four areas.
The enrichment of the rich economy is dependent on the poor economy. The richer is the poor economy, the stronger is its positive spillover effect. Figure 8 shows that this positive spillover effect prevails and enriches the rich economy if the poor economy’s capital stock is above $B$ while the rich economy slows down below $B$. The direction of arrows combines the overall effect. Following the directions of the arrow, the capital stock of the two economies would cycle around the unstable steady state $S$.

The above argument can be confirmed by looking at Figure 9 (a) and (b). They show the limiting behavior of the system just after the bifurcation point at $L = 0.175$ as time series of $k^1$ and $k^2$ and as attractor plots in the state space. Figure 9 (b) shows the loss of stability of the asymmetric steady state and the emergence of a closed invariant curve indicating that the bifurcation is supercritical (see Kuznetsov (1998) for details).

The second bifurcation point is at $L_{bb}$ where the steady state value of $k^1$ reaches $K(\lambda) = 2.89$ (See Figure 7). At this point the dynamical system is not differentiable, as it switches from the case in which both economies face the borrowing constraint to the
case in which one economy faces the borrowing constraint. It is shown in the proof of Proposition 6 that the asymmetric steady state where \( k^2 < K(\lambda) < k^1 \) is stable, while the asymmetric steady state where \((k^1, k^2) < (K(\lambda), K(\lambda))\) is unstable, around this bifurcation point. This implies that there are two forces pulling the world economy in opposite directions. One pulls the economy close to the former steady state while the other pulls the economy away from it. These two forces generate non-stationary orbits of \( k^1_t \) around \( K(\lambda) \).

6 Concluding Remarks

We have examined how the stability of the two country model is influenced by the spillover effect through the world interest rate. We singled out the spillover effect by comparing the Matsuyama model to the two country model. The symmetry breaking results in Boyd & Smith (1997) and Matsuyama (2004) hold in the present paper. There are some additional common features in Boyd & Smith (1997), Matsuyama (2004) and the present paper. Firstly, an initially poor country remains relatively poor if it does not converge to the symmetric steady state. Secondly, the poor country is better off in the symmetric steady state than in asymmetric steady states, while the rich country is worse off. Thirdly, the aggregate wealth of the world economy is higher in the symmetric steady state than in asymmetric steady states. These results are the consequence of the borrowing constraint which limits the amount of domestic investment in the poor country forcing it to supply credit in the international financial market.

In addition, the present paper shows that a new stable asymmetric steady state arises in the presence of the spillover effect when the relative population size is different in the two economies. This steady state undergoes a bifurcation generating endogenous fluctuations of international financial flows between the poor and the rich economy. It is surprising that endogenous cycles can arise in a relatively simple model by simply giving weights to two economies (two “large” economies with a different population size). It is natural to suspect that there are other models which exhibit endogenous cycles caused by the interaction of two economies which are integrated through a financial market. Further investigations into the structures and interaction mechanisms of economies will provide new insights into the relationship between the inequality and the fluctuation of income embodied in international economic development.
Appendix

Preliminaries

This section provides the properties of the graph of Eqs. (6), (7) and (8) which are used below for the proof of Propositions 1, 2 and 5.

Lemma 1. The level set defined by $G(k^1, k^2) = 0$ has the configuration depicted in Figure 10 (a). Moreover, $\frac{dk^2}{dk^1} = -1$ at $G(K^*(R), K^*(R)) = 0$.

Obviously, $(0, 0), (K^*(R), 0), (0, K^*(R))$ and $(K^*(R), K^*(R))$ are solutions for $G(k^1, k^2) = 0$. For $L = 1/2$, totally differentiating Eq.(6) we obtain $(1-RW'(k^1)) + (1-RW'(k^2)) \frac{dk^2}{dk^1} = 0 \Leftrightarrow (1-RW'(k^1))(1-RW'(k^2)) + (1-RW'(k^2))^2 \frac{dk^2}{dk^1} = 0$. Therefore,

$$\frac{dk^2}{dk^1} < (>) 0 \Leftrightarrow (1-RW'(k^1))(1-RW'(k^2)) > (<) 0.$$

From this we obtain Figure 10 (a).

Lemma 2. The level set defined by $H(k^1) - H(k^2) = 0$ has the configuration depicted in Figure 10 (b). Moreover, $\frac{dk^2}{dk^1} = -1$ at $H(K^*(R)) - H(K^*(R)) = 0$.

Obviously, if $k^1 = k^2$, Eq.(7) is satisfied. Differentiating $H(k)$ we obtain

$$H'(k) \equiv \frac{\lambda f''(k)(1 - f(k))}{(1 - W(k))^2} > (<) 0 \Leftrightarrow f(k) > (<) 1. \quad \text{(A.1)}$$

Totally differentiating $H(k^1) - H(k^2) = 0$ we obtain $H'(k^1) - H'(k^2) \frac{dk^2}{dk^1} = 0 \Leftrightarrow H'(k^1)H'(k^2) - (H'(k^2))^2 \frac{dk^2}{dk^1} = 0$. Therefore, $\frac{dk^2}{dk^1} > (<) 0 \Leftrightarrow H'(k^1)H'(k^2) > (<) 0$. It follows that

$$\frac{dk^2}{dk^1} > 0 \Leftrightarrow f(k^1), f(k^2) > 1 \text{ or } f(k^1), f(k^2) < 1. \quad \text{(A.2)}$$

Eqs.(A.1) and (A.2) imply $\frac{dk^2}{dk^1} > 0$ if and only if $k^1 = k^2$ and $\frac{dk^2}{dk^1} < 0$ if and only if $f(k^1) < 1 < f(k^2)$ or $f(k^2) < 1 < f(k^1)$. From this we obtain Figure 10 (b).

Lemma 3. The function $\phi$ has the configuration depicted in Figure 11 for $\lambda < \alpha$ and

1. is monotonically increasing for $k \in [0, K(\lambda)]$ if $\lambda \geq \alpha$,
2. has a unique inflection point at $k = \left(\frac{1+\alpha}{\lambda}\right)^{\frac{1}{\alpha}}$,
3. is concave for $k \in [0, K(\lambda))$ if $\phi'(K(\lambda)) > -1$. 

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\[ G(k^1, k^2) = 0 \]

\[ H(k^1) = H(k^2) \]

Figure 10: Steady state conditions when \((k^1, k^2) < (K(\lambda), K(\lambda))\)

\[ \phi(k) \]

Figure 11: The graph of \(\phi(k)\)

Eq. (8) defines \(k^2\) as a function of \(k^1\) for \(k_1 \in [0, K(\lambda))\). The function \(\phi\) satisfies \(\phi(0) = 0\) and \(\phi(K(\lambda)) = K(\lambda)\). Furthermore,

\[
\phi'(k) = \frac{\alpha}{1-\alpha} \frac{A}{H(k)} f''(k) \left( \frac{f(k)-1}{1-W(k)f'(k)} \right)
\]

\[
\phi''(k) = \frac{k}{\phi'(k)} \left( \frac{(1-(1-\alpha)W(k)\alpha f(k)-(1-\alpha)(1-W(k))^2}{f(k)-1(1-W(k))} - \frac{f(k)-1}{1-w(k)} - (2-\alpha) \right) . \tag{A.3}
\]

1) From Eq. (A.3), we obtain that \(\phi\) is monotonically increasing for \(k \in [0, K(\lambda))\) if \(f(K(\lambda)) \leq 1\), which is equivalent to \(\lambda \geq \alpha\).

2) From Eq. (A.3), \(\phi''(k) = 0\) is equivalent to \(k = \left(\frac{1+\alpha}{A}\right)^{\frac{1}{\alpha}}\). This implies that the function \(\phi\) has a unique inflection point.

3) Suppose that \(\phi''(k) > 0\). Then, \(k > \left(\frac{1+\alpha}{A}\right)^{\frac{1}{\alpha}}\). This and \(k < K(\lambda)\) imply that

\[
\left(\frac{1+\alpha}{A}\right)^{\frac{1}{\alpha}} < K(\lambda) \Leftrightarrow \lambda < \alpha^2 . \tag{A.4}
\]
This means that the inflection point of $\phi(k)$ is at $k < K(\lambda)$ if and only if $\lambda < \alpha^2$. Our assumption was $\phi'(K(\lambda)) > -1$, which is equivalent to $\lambda > \frac{\alpha}{2-\alpha}$. This and (A.4) together imply $\alpha - \frac{1}{2-\alpha} > 0$, which is never satisfied for $\alpha \in (0,1)$. Hence, $\phi(k)$ is concave for $k \in [0, K(\lambda))$ if $\phi'(K(\lambda)) > -1$. □

Proof of Proposition 1

If $k_1, k_2 < K(\lambda)$, the steady state conditions are Eqs.(6) and (7). From Lemmas 1 and 2 the necessary condition for the asymmetric steady states $(k_1, k_2) < (K(\lambda), K(\lambda))$ to exist is $K^*(R_c) < K(\lambda)$, which is equivalent to $\lambda < \alpha$. The sufficient condition is $\lambda < \alpha$ and $R > R_c$ (See Figure 12).

Figure 12: Existence of asymmetric steady states when $k_1, k_2 < K(\lambda)$

If $k_1 < K(\lambda) \leq k_2$, the steady state conditions are Eqs.(6) and (8). From Lemma 3 we know that $\phi$ is monotonically increasing for $k \in [0, K(\lambda))$ if $\lambda \geq \alpha$. Moreover, $\phi(K(\lambda)) = K(\lambda)$. Therefore, there exist no intersection of the graph of Eqs.(6) and (8) if $\lambda \geq \alpha$. □

Proof of Proposition 2

If $k_1, k_2 < K(\lambda)$, the steady state conditions are Eqs.(6) and (7). From the proof of Proposition 1 we know that Eq.(6) and Eq.(7) have two intersections if and only if $R > R_c$ and $\lambda < \alpha$. Without loss of generality, suppose that $k_2 > k_1$. If $k_2$ is equal to $K(\lambda)$, Eq.(7) is identical to Eq.(8). This implies the continuity of asymmetric steady states in $R$. If $k_1 < K(\lambda) \leq k_2$, the steady state conditions are Eqs.(6) and (8).
Note that \( \lambda \geq \alpha \) is equivalent to \( \phi'(K(\lambda)) \geq 0 \) implying that there exist no asymmetric steady states (Proposition [L]). Therefore, we examine the case where \( \phi'(K(\lambda)) < 0 \). Below we examine the number of intersections of the graph of \( \phi(k^1) \) and \( G(k^1, k^2) = 0 \)

When \(-1 < \phi'(K(\lambda)) < 0\) and when \( \phi'(K(\lambda)) < -1\). Figure [13] helps to provide a geometric interpretation of the proof.

Suppose that \(-1 < \phi'(K(\lambda)) < 0\). Remember that \( \frac{dk^2}{dk^1} = -1 \) at \( G(K(\lambda), K(\lambda)) = 0 \) (Lemma [L]). Then, the concavity of \( \phi(k) \) for \( k \in [0, K(\lambda)] \) (Lemma [3]) ensures one intersection for \( R_c < R \leq (K^*)^{-1}(K(\lambda)) \). The concavity of both \( \phi(k^1) \) and \( G(k^1, k^2) = 0 \) for \( k^1 \in [0, K(\lambda)] \) ensures two intersections for \( (K^*)^{-1}(K(\lambda)) < R < R_{cc} \) and excludes a third intersection. For \( R > R_{cc} \) there are no intersections of the graph of \( \phi(k^1) \) and \( G(k^1, k^2) = 0 \).

Suppose that \( \phi'(K(\lambda)) \leq -1 \). Since \( \frac{dk^2}{dk^1} = -1 \) at \( G(K(\lambda), K(\lambda)) = 0 \), \( G(k^1, k^2) = 0 \) has a unique intersection with the graph of \( \phi(k^2) \) for \( R_c < R \leq (K^*)^{-1}(K(\lambda)) \) and two intersections for \( (K^*)^{-1}(K(\lambda)) < R < R_{cc} \). For \( R > R_{cc} \) there is no intersection of the graph of \( \phi(k^1) \) and \( G(k^1, k^2) = 0 \). For a third intersection to exist, the graph \( \phi(k^1) \) has to cut \( G(k^1, k^2) = 0 \) from inside at the third intersection. This would mean that \( \phi''(k^1) \) has to change its sign two times in \( [0, K(\lambda)] \). This is a contradiction because we know that \( \phi(k) \) has a unique inflection point (Lemma [3]). Hence, \( \phi'(K(\lambda)) \leq -1 \) guarantees that there are no more than two intersections. \( \Box \)
Proof of Proposition 3

1) Let $\Psi^i(k^1, k^2) \equiv \Psi(k^i, R(k^1, k^2))$ for $i = 1, 2$. If $K^*(R) < K(\lambda)$, the local dynamics around the symmetric steady state is characterized by

$$k^1 = \Psi^1(k^1, k^2) \equiv \frac{R [W(k^1) + W(k^2)]}{1 + \left[ \frac{1 - W(k^2)}{1 - W(k^1)} \right]^{\alpha - 1}},$$

$$k^2 = \Psi^2(k^1, k^2) \equiv \frac{R [W(k^1) + W(k^2)]}{1 + \left[ \frac{1 - W(k^1)}{1 - W(k^2)} \right]^{\alpha - 1}}.$$

Let

$$J(k, k) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial \Psi^1(k,k)}{\partial k^1} & \frac{\partial \Psi^1(k,k)}{\partial k^2} \\ \frac{\partial \Psi^2(k,k)}{\partial k^1} & \frac{\partial \Psi^2(k,k)}{\partial k^2} \end{pmatrix},$$

where $k = K^*(R) \equiv \left( \frac{1}{(1-\alpha)AK} \right)^{\frac{1}{\alpha - 1}}$. Observe that $a = d, b = c$. The characteristic polynomial reads $p(\mu) = \mu^2 - 2\alpha \mu + a^2 - b^2$. The eigenvalues of the system are $\mu_1 = a + b = \alpha$ and $\mu_2 = a - b = \alpha \left( \frac{Ak^\alpha}{1-(1-\alpha)Ak^\alpha} \right)$. It follows that $0 < \mu_1 < 1$ and $0 < \mu_2 < 1$ if and only if $R < R_c = \frac{1}{(1-\alpha)AK^*}$.

If $K^*(R) \geq K(\lambda)$, the local dynamics around the symmetric steady state is characterized by

$$k^1 = \Psi(k^1, R(k^1, k^2)),$$

$$k^2 = \Psi(k^1, R(k^1, k^2)).$$

This implies if $k^1_0, k^2_0 \geq K(\lambda), k^1_0 = k^2_0 = k_1 > 0$. In general, if $k^1_0 = k^2_0 = k_0 > 0$, then $\Psi^1(k_0, k_0) = \Psi^2(k_0, k_0) = RW(k_0) = k_1 = k^2_1 = k^1_2 = k_1 > 0$. By induction, $(\Psi^1)^n(k_0, k_0) = (\Psi^2)^n(k_0, k_0) = \underbrace{RW \circ RW \circ \cdots \circ RW}_{n\text{-times}}(k_0) = k_1^n = k_2^n = k_n > 0, \forall n \in \mathbb{N}.$

Given Assumption 1, the orbit $\lim_{n \to \infty} \underbrace{RW \circ RW \circ \cdots \circ RW}_{n\text{-times}}(k_0)$ converges to $K^*(R)$ and hence $(k_1^n, k_2^n)$ converges to the symmetric steady state $(K^*(R), K^*(R))$. It follows that if $K^*(R) > K(\lambda)$, the symmetric steady state is locally stable.

2) If $R_c \leq R \leq (K^*)^{-1}(K(\lambda))$, the symmetric steady state is locally unstable from part 1) of this proof. However, part 1) also shows that there exists a saddle path $k^1 = k^2$ leading to this symmetric steady state. □
Proof of Proposition 4

The proof follows directly from part 1) of the proof of Proposition 3. □

Proof of Proposition 5

Totally differentiating Eq.(6) we obtain \( \frac{dk_2^2}{dk_1} = -\frac{L}{1-L} \) at \( k_1 = k_2 = K^*(R) \) (Lemma 1). Figure 14 depicts how \( G(k_1, k_2) = 0 \) is affected by a change in \( L \). Notice that \( H(k_1) = H(k_2) \) is not affected by a change in \( L \). There exists always a unique value of \( L \) at which \( H(k_1) = H(k_2) \) is tangent to \( G(k_1, k_2) = 0 \) (Lemmas 1 and 2). □

Figure 14: Broken symmetric structure

Proof of Proposition 6

We will prove numerically that the asymmetric steady state \( (k_1, k_2) < (K(\lambda), K(\lambda)) \) undergoes a supercritical Neimark-Sacker bifurcation by calculating the determinant and trace. The determinant and the trace of the Jacobian matrix of the system (5) when \( k_1, k_2 < K(\lambda) \) can be written as

\[
\det = \frac{W'(k_1)k_1W'(k_2)k_2^2}{(LW(k_1)+(1-L)W(k_2))^2} R(1-\alpha) \left( k_2^2 L^2 \frac{(1-W(k_1))^{\frac{1}{1-\alpha}}}{(1-W(k_2))^{\frac{1}{1-\alpha}}} + k_1(1-L)^2 \frac{(1-W(k_2))^{\frac{1}{1-\alpha}}}{(1-W(k_1))^{\frac{1}{1-\alpha}}} \right. \\
+ \left. k_1^2(1-L)L \frac{(1-W(k_1))^{\frac{1}{1-\alpha}}}{(1-W(k_2))^{\frac{1-\alpha}{1-\alpha}}} + k_1^2(1-L)L \frac{(1-W(k_2))^{\frac{1-\alpha}{1-\alpha}}}{(1-W(k_1))^{\frac{1-\alpha}{1-\alpha}}} \right) (A.5)
\]

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and

\[
\text{tr} = \frac{1}{L W(k^1) + (1-L)W(k^2)} \left( W''(k^1)k^1 \left( L + \frac{k^1(1-L)(1-W(k^2))\lambda^{\frac{1}{1-\alpha}}}{R(1-\alpha)} \right) + W''(k^2)k^2 \left( 1 - L + \frac{k^2L}{R(1-\alpha)} \frac{(1-W(k^1))\lambda^{\frac{1}{1-\alpha}}}{(1-W(k^2))\lambda^{\frac{1}{1-\alpha}}} \right) \right). \tag{A.6}
\]

The determinant and the trace when \(k^2 < K(\lambda) < k^1\) can be written as

\[
\det = \frac{W'(k^1)k^1W'(k^2)k^2}{(LW(k^1) + (1-L)W(k^2))^2} \left( k^2L \frac{\lambda^{\frac{1}{1-\alpha}}}{(1-W(k^2))\lambda^{\frac{1}{1-\alpha}}} + k^1(1 - L) \frac{1-W(k^2)(\lambda^{\frac{1}{1-\alpha}})}{\lambda^{\frac{1}{1-\alpha}}} \right) \tag{A.7}
\]

and

\[
\text{tr} = \frac{W'(k^1)k^1L + W'(k^2)k^2}{LW(k^1) + (1-L)W(k^2)} \left( 1 - L + \frac{k^2L}{R(1-\alpha)} \frac{(1-W(k^1))\lambda^{\frac{1}{1-\alpha}}}{(1-W(k^2))\lambda^{\frac{1}{1-\alpha}}} \right). \tag{A.8}
\]

Figure 15 shows how the determinant and the trace of the system move as we change the bifurcation parameter \(L\). The points (a),(b),(c),(d),(e),(f) correspond to \(L = (0.117, 0.117, 0.13, 0.16, 0.177, 0.19)\). The point (a) is defined by Eqs.(A.7) and (A.8), and the points (b)-(f) by Eqs.(A.5) and (A.6). We observe that at \(L = 0.117\) when \(k^1 = K(\lambda)\), the determinant and the trace jump from (a) to (b). Comparing Eqs.(A.5) and (A.6) with Eqs.(A.7) and (A.8), we see that the determinant and the trace are not equal respectively at \(k^1 = K(\lambda)\). The dynamical system is not differentiable at this point. As the value of \(L\) increases, the determinant crosses 1 at the point (e) which proves that the bifurcation is a Neimark-Sacker bifurcation. That it is a supercritical bifurcation can be confirmed by the observation of the closed invariant curve in Figure 9 (b).
References


