ON THE ROLE OF ASSETS FOR THE DYNAMICS OF CAPITAL ACCUMULATION *

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Abstract

The paper examines the effect of an asset market in a standard Overlapping Generations model which includes an exogenous process with a positive yield, a so called ‘Lucas tree’. It compares the dynamics under rational expectations of the economy with and without an asset market. It is shown that the asset market generates multiple steady states (poverty trap) when the capital share is greater than one half. In contrast, the economy can achieve Pareto improvements through the asset market when the capital share is less than one half.

Keywords: asset market, rational expectations equilibrium, poverty trap, Pareto improvement

JEL classification: C62, G11, O11

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1 Introduction

Since the studies of Goldsmith (1969), McKinnon (1973), and Shaw (1973) the connection between the financial sector and the real sector is a well established fact. The links that are usually emphasized are the role of financial markets in channeling savings towards more productive investment.\footnote{See Greenwood & Jovanovic (1990), Saint-Paul (1992), Bencivenga & Smith (1991), Zilibotti (1994), Greenwood & Smith (1997), and Acemoglu & Zilibotti (1997) for the role of financial intermediaries on growth in a general equilibrium framework.} Even though financial markets promote growth in different ways, there might be a cost involved in opening the financial market. The role of fixed costs when opening new markets is emphasized in Greenwood & Jovanovic (1990), Saint-Paul (1992), Zilibotti (1994), and Greenwood & Smith (1997). In these papers, there is a “threshold effect”, i.e. if the economy is wealthy enough to open a financial market then it increases the equilibrium rate of growth. Saint-Paul (1992) was the first to show that multiple equilibria may arise in the presence of a strategic complementarity between financial markets and technology. Similarly, multiple equilibria may arise due to a strategic complementarity in the investment demands of final producers (See Zilibotti (1994)).

In Saint-Paul (1992) a more developed financial market allows for a more specialized technology and thus leads to higher growth. This implies that there is a strategic complementarity between financial markets and technology because both can be used for risk diversification. Because of the fixed cost, however, individuals may be reluctant to open a financial market if their income is too low. In Zilibotti (1994) each firm, when investing, adds to the total demand of intermediation services and causes a fall in the gap between the price of internal and external capital. Firms ignore such external effects and cannot coordinate their demand. Without the coordination of demand, the intermediation cost is too high and people consume more now than in the future and the economy converges to a zero equilibrium.

The present paper is related to the above literature on financial development and growth but takes an alternative approach. The literature on financial development and growth assumes that more financial intermediation contributes to higher growth. Hence, the literature focuses on malfunctioning of financial markets as an impediment to growth. The aim of the present paper is to question this premise. We employ a standard Overlapping Generations (OLG) model which is extended to include an exogenous technology
often referred to as a “Lucas’ tree”. This exogenous technology yields constant proceeds every period and may be interpreted as natural resources available to the economy at no costs or simply as an apple tree growing next to the neoclassical firm. Suppose that before markets are financially developed, in each period old generations can harvest the tree but they can not sell the ownership of the tree and therefore pass it on to younger generations as bequest. Let us assume that “financial development” means that the ownership of the tree can be traded in a competitive asset market.

The model may be compared with the OLG models with Fiat money (ex. Wallace (1980) and Tirol (1985)). However, the paper asset in our model is not intrinsically useless (see Wallace (1980)) since it entitles the owner to the products from the “tree”. It is shown that the economy converges to a unique equilibrium irrespective of initial capital stocks without the asset market. However, once the asset market is opened we observe two contrasting results. On the one hand, if the capital share in production is more than one half, multiple equilibria may arise. In this case the economy may suffer from persistent underdevelopment if the initial capital stock is below the threshold value. This happens because the purchase of assets diverts part of the savings from capital investment and agents do not invest enough capital for production to sustain the capital stock of the economy. On the other hand, if the capital share in the production is less than one half, the economy may achieve a Pareto improvement by opening the asset market, which mitigates the overaccumulation of capital in the economy without the asset market.

The paper is organized as follows. Section 2 introduces the basic model and Section 3.1 introduces the asset market into the economy. Section 3.2 characterizes the rational expectations equilibria. Section 3.3 derives the minimum state variable solution to analyze the dynamics. Section 3.4 discusses welfare issues in the economy with and without the asset market. Section 4 concludes.

\footnote{Tobin (1965) was one of the first to present the idea that holding money to store value may lead to reduction of capital accumulation. See also Wallace (1980) for the role of Fiat money in the overlapping generations model.}

\footnote{This equilibrium can be compared to the “bubbly” equilibrium in Tirol (1985), in which money has a positive value in spite of the fact that it is intrinsically useless. The bubbly equilibrium is associated with a positive price and the golden rule of capital accumulation. In our paper, the paper asset is not intrinsically useless as it entitles owners to consume the exogenous production.}
2 The Basic Model

The economy evolves in discrete time and consists of markets for output, labor, and capital. There exists a neoclassical firm which produces output using capital and labor. Production in per capita terms is described by a function \( f: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \). The firm maximizes profit every period and pays wage and return on capital according to the marginal product rule, i.e.

\[
  r(k) := f'(k) \quad \text{and} \quad w(k) := f(k) - kf'(k).
\]  

(1)

\[ \text{Assumption 1} \quad \text{The production function } f: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ is } C^2, \text{ strictly increasing, strictly concave, and satisfies the Inada conditions with } f(0) = 0. \text{ Furthermore, } w'(0) = \infty \text{ and } w''(k) < 0. \]

It is well known that in an OLG framework, multiplicity of equilibria can arise due to the non concavity of the wage function. Therefore, the second condition is imposed in order to avoid any multiplicity arising from the technology alone. For simplicity it is assumed that capital depreciates fully in each period. Then, next period’s capital stock is determined only by new capital investment.

There are overlapping generations of consumers who live for two consecutive periods. Therefore, in any period, there are always two generations alive referred to as the young and the old. For simplicity we assume no population growth. A young consumer of generation \( t \) supplies one unit of labor inelastically to the labor market in the first period of his lifetime and receives labor income \( w(k) \). His lifetime utility depends on old age consumption only.

Suppose that there exists an exogenous production process, a so called “Lucas’ tree” in addition to the neoclassical firm, which yields \( d > 0 \) units of consumption goods in each period. Let us assume that old consumers own the “tree” and pass its ownership on to the next generation as bequest. Since there is no intertemporal consumption decision, the law of motion for capital is simply

\[
k_1 = w(k)\]

(2)

4To minimize notation we suppress time indices whenever possible. Variables without time subscript refer to an arbitrary time period \( t \), while the subscript 1 refers to \( t + 1 \).
Given Assumption 1, equation (2) implies that the economy has a unique positive globally stable steady state, \( k^* \) which is defined by the solution of \( k = w(k) \). Equilibrium consumption of the old is
\[
c^* = k^*r(k^*) + d. \tag{3}
\]

3 The Economy with an Asset Market

3.1 The Asset Market

Let us now introduce a market where the old can sell the ownership of the ‘tree’ to the young. The ownership entitles the owner to receive the exogenous output in the future. Assume that the ownership can be traded in the form of a paper asset, whose supply is constant over time and normalized to unity. In this case, the young agent may choose between real capital investment and the paper asset and faces the budget constraint
\[
s + xp \leq w(k). \tag{4}
\]
s denotes the amount of investment in physical capital and \( x \) is the number of paper assets purchased at price \( p \). The old agent’s budget constraint is
\[
c_1 \leq sr_1 + x(p_1 + d), \tag{5}
\]
where \( r_1 \) is next period’s return on capital, \( p_1 \) is next period’s selling price of the paper asset and \( d \) may now be interpreted as the amount of dividend received on the asset holdings. When the young makes the portfolio decision \((x, s), (r_1, p_1)\) are unknown variables. Let us assume that the young makes point forecasts \((r^e, p^e)\) for both variables. Then, for given values of \((w(k), r^e, p^e, p) \geq 0\), the young agent’s objective is to maximize old age consumption given the current budget constraint
\[
B(w(k), p) = \{x| x \geq 0, xp \leq w(k)\}. \tag{6}
\]
Equations (4), (5) and (6) imply that demand of the paper asset \( \varphi(w(k), r^e, p^e, p) \) is given by the correspondence
\[
\varphi(w(k), r^e, p^e, p) = \begin{cases} 
0 & \text{if } pr^e > p^e + d \\
x \in \left[0, \frac{w(k)}{p}\right] & \text{if } pr^e = p^e + d \\
\frac{w(k)}{p} & \text{if } pr^e < p^e + d.
\end{cases} \tag{7}
\]
3.2 Rational Expectations Equilibrium

Since the amount of the paper assets in the economy is normalized to one, a market clearing price $p$ must satisfy
\[ \varphi(w(k), r^e, p^e, p) = 1. \]

**Proposition 1** For a given pair $(r^e, p^e) \gg 0$, the unique price clearing the asset market is given by
\[ p = S(r^e, p^e) := \frac{p^e + d}{r^e}. \]

**Proof:** Given $(r^e, p^e) \gg 0$, suppose $0 < pr^e < p^e + d$. The specific demographic structure of the model implies that all assets sold by old consumers are bought by the young. This contradicts rational expectations of next period’s capital return, since zero capital investment implies an infinite return on capital next period. Conversely, suppose $0 < p^e + d < pr^e$. Then, $p$ can not be a market clearing price because the asset demand is zero and the supply is unity. \( \square \)

Since all the assets sold by the old are bought by the young, the law of motion for capital is
\[ k_1 = A(k, p) := w(k) - p. \]

### 3.2 Rational Expectations Equilibrium

**Definition 1** A Rational Expectations Equilibrium (REE) is a pair $(k, p) \in \mathbb{R}^2_+$, such that

a) given $k \in \mathbb{R}_+$, the asset price $p \in \mathbb{R}_+$ is a fixed point of the price law, $p = S(r(k), p)$ and

b) given $p \in \mathbb{R}_+$, the capital stock is a fixed point of the capital accumulation equation, $k = A(k, p)$.

The above definition implies that a pair $(k, p) \geq 0$ is a REE if it satisfies the system of equations
\[ k = w(k) - p \quad \text{and} \quad p = \frac{p + d}{r(k)}. \]

Eliminating $p$ yields
\[ w(k) - k = \frac{d}{r(k) - 1}. \]
To obtain $p \geq 0$, equation (11) implies that $k$ is restricted to the interval $[0, \bar{k}]$ where $ar{k} = \min\{\bar{k}_1, \bar{k}_2\}$ and $\bar{k}_1$ and $\bar{k}_2$ are solutions of equations $w(k) = k$ and $r(k) = 1$ respectively. Outside $[0, \bar{k}]$ either $w(k) < k$ or $r(k) < 1$ holds and the asset price is negative.

Observe that $(0, 0)$ is always an equilibrium. When $k = 0$, agents do not demand assets and the asset price is zero. When $p = 0$, $k = 0$ is a fixed point of the capital accumulation equation. Rearranging equation (11), an interior REE should satisfy the equation $g(k) = h(k)$ where $g(k) := w(k)r(k)$ and $h(k) := f(k) - k + d$. Notice that $h$ is increasing on the interval $[0, \bar{k}]$ with $h(0) = d$ and $g(\bar{k}) < h(\bar{k})$.

**Proposition 2** Let the production function be $f(k) := Ak^\theta$ of the Cobb-Douglas form and $n$ be the number of REE.

a) If $\theta < 0.5$, then $n = 2$.

b) If $\theta = 0.5$, then $n = 1$ for $A^2 \leq 4d$ and $n = 2$ for $A^2 > 4d$.

c) If $\theta > 0.5$, then $n = 1$ for $A < A^*(\theta, d)$, $n = 2$ for $A = A^*(\theta, d)$ and $n = 3$ for $A > A^*(\theta, d)$,

where $A^*(\theta, d)$ denotes the value of $A$ for which the functions $g(k)$ and $h(k)$ are tangent to each other.

**Proof:** Figure[i] characterizes the proof geometrically.

a) If $\theta < 0.5$, $g(0) = \infty$ and $g'(k) < 0$. Since $h'(k) > 0$, $h(0) = d$ and $g(\bar{k}) < h(\bar{k})$, the claim follows.

b) If $\theta = 0.5$, $g(k) = \frac{A^2}{4}$. Since $h'(k) > 0$ and $h(0) = d$, the claim follows.

c) If $\theta > 0.5$, $g(0) = 0$ and $g'(k) > 0$. On the other hand $h'(k) > 0$, $h(0) = d$ and $g(\bar{k}) < h(\bar{k})$. For $A < A^*(\theta, d)$, equation $g(k) = h(k)$ does not have a solution, for $A = A^*(\theta, d)$ it has one solution and for $A > A^*(\theta, d)$, it has two solutions.

\[\square\]

\[\text{Assumption} \text{i}\] guarantees the existence and uniqueness of $\bar{k}_1$ and $\bar{k}_2$. 

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3.2 Rational Expectations Equilibrium
3.3 Dynamics Under Rational Expectations

To describe the dynamics under rational expectations we apply the so called minimum state variable (MSV) solution. From a dynamical systems point of view this corresponds to the associated functional rational expectations equilibrium. The MSV solution is defined by a pair of price and capital accumulation functions \((P, G)\), \(P : \mathbb{R}_+ \to \mathbb{R}_+\) and \(G : \mathbb{R}_+ \to \mathbb{R}_+\) such that for any \(k \geq 0\) the following system of equations is satisfied.

\[
\begin{align*}
G(k) &= w(k) - P(k) \quad \text{(12)} \\
\frac{d}{dk} \left( P(G(k)) \right) &= \frac{P(G(k)) + d}{r(G(k))}. \quad \text{(13)}
\end{align*}
\]

**Proposition 3** The system of functional equations (12) and (13) has a unique continuous solution.

**Proof:** See the appendix. \(\square\)

In order to analyze the dynamics under rational expectations we establish some properties of the function \(G\).

**Proposition 4** \(G(0) = 0, \ G(\infty) = \infty\), \(G\) is monotonically increasing and

\[
\lim_{k \to \infty} \frac{G(k)}{k} = 0.
\]

**Proof:** Since \(k = 0\) is always a REE it follows that \(G(0) = 0\). Suppose that \(G(\infty) = C < \infty\). Then, equation (12) implies that \(P(\infty) = \infty\) while equation (13) implies that

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\( P(\infty) = \frac{P(C)d}{r(C)} < \infty \), which is a contradiction. Therefore, \( G(\infty) = \infty \). Moreover, \( \lim_{k \to \infty} \frac{G(k)}{k} \leq \lim_{k \to \infty} \frac{w(k)}{k} = 0 \). In order to show that \( G \) is monotonically increasing we first show that \( G \) is a bijective map. Let us start by contradiction. Suppose that there exists \( k_1 \neq k_2 \) such that \( G(k_1) = G(k_2) \). Then equation (13) implies that \( P(k_1) = P(k_2) \) and (13) implies \( w(k_1) = w(k_2) \), which contradicts monotonicity of the wage function. Since \( G(0) = 0 \) and \( G(\infty) = \infty \) it follows that \( G \) is monotonically increasing. \( \square \)

**Proposition 5** Let the production function be \( f(k) := Ak^\theta \) of the Cobb-Douglas form.

a) If \( \theta < 0.5 \), then \( G'(0) = \infty \).

b) If \( \theta = 0.5 \), then \( G'(0) = \frac{A^2}{4d} \).

c) If \( \theta > 0.5 \), then \( G'(0) = 0 \).

**Proof:** Equation (13) implies that \( P(k) = O([G(k])^{1-\theta}) \) as \( k \to 0 \). Together with equation (12) this implies that \( O([G(k)]^{1-\theta}) + G(k) = w(k) \). It follows that \( G(k) = O([w(k)]^{\frac{\theta}{1-\theta}}) \) and consequently \( G(k) = O(k^{\frac{\theta}{1-\theta}}) \) as \( k \to 0 \). This implies that the asymptotic of \( G'(k) \) is

\[
G'(k) = O(k^{\frac{2\theta}{1-\theta}}) \quad \text{as} \quad k \to 0. \tag{15}
\]

Therefore,

\[
G'(0) = \begin{cases} 
\infty & \text{if } \theta < 0.5 \\
0 & \text{if } \theta > 0.5.
\end{cases} \tag{16}
\]

When \( \theta = 0.5 \), \( G(k) = O(k) \) and \( G'(k) = O(1) \) as \( k \to 0 \). Suppose that \( G'(0) = C \). Then, for sufficiently small \( k \), \( G(k) = Ck \) and \( P(k) = w(k) - Ck \). Together with equations (12) and (13) this implies that for a sufficiently small \( k \), \( C \) should satisfy the equation

\[
(w(k) - Ck)r(Ck) = w(Ck) - C^2k + d. \tag{17}
\]

For a sufficiently small \( k \), equation (17) can be rewritten as

\[
\frac{A\sqrt{k}}{2} \frac{A}{2\sqrt{Ck}} = d, \tag{18}
\]

which implies that \( C = \frac{A^2}{4d} \).

\footnote{\( f(k) = O(k^\delta) \) as \( k \to 0 \) if and only if \( \exists \delta > 0, \exists C > 0 \) such that \( |f(k)| \leq C|k^\delta| \) for \( |k| < \delta \).}
Using the results from Propositions 2, 4 and 5 we can characterize the dynamics of the economy.

**Proposition 6** Let the production function be \( f(k) := Ak^\theta \) of the Cobb-Douglas form.

a) If \( \theta < 0.5 \), then the unique interior REE is stable.

b) If \( \theta = 0.5 \), then

- the unique interior REE is globally stable for \( A^2 > 4d \).
- the unique zero REE is globally stable for \( A^2 \leq 4d \).

c) If \( \theta > 0.5 \), then

- for \( A < A^*(\theta, d) \) the unique zero REE is globally stable.
- for \( A = A^*(\theta, d) \) there exist an interior REE and the zero REE, which are both asymptotically stable.
- for \( A > A^*(\theta, d) \) there exist two interior REE and the higher REE and the zero REE are asymptotically stable while the lower REE is unstable.

Properties a) - c) follow directly from Propositions 2, 4 and 5. Figure 2 characterizes the different cases geometrically.

![Figure 2: Time one map G(k)](image)

Figure 2 (b) and (c) show that the asset market may create a poverty trap if \( \theta \geq 0.5 \) and if the productivity parameter \( A \) is too low. The economy is caught in a downward
spiral and suffers from persistent underdevelopment converging to the zero steady state. If \( \theta > 0.5 \), even for sufficiently high values of \( A \) the economy can not escape from the poverty trap if the initial capital stock is below the critical threshold.

### 3.4 The Asset Market and Welfare

The aim of this section is to compare the welfare of the economy with and without the asset market. In this section variables denoted with superscript \( a \) refer to an economy with the asset market and variables denoted with superscript \( b \) refer to an economy without the asset market. Then, equilibrium consumption levels are

\[
\begin{align*}
  c^b &:= \theta f(k^b) + d = f(k^b) - k^b + d \\
  c^a &:= \theta f(k^a) + P(k^a) + d = f(k^a) - k^a + d.
\end{align*}
\]

Equilibrium consumption is maximized if \( f'(k) = 1 \). This is the case when the young invests the capital income and the old consumes the wage income plus the exogenous product. However, in the economy without the asset market, the young invests the wage income and the old consumes the capital income plus the exogenous product. This implies that there is overaccumulation of capital if \( \theta < 1/2 \). If \( \theta > 1/2 \), there is underaccumulation of capital. In contrast, there is always underaccumulation of capital in the economy with an asset market since \( f'(k^a) > 1 \) holds for any REE.

**Proposition 7** Let the production function be \( f(k) := Ak^\theta \) of the Cobb-Douglas form.

1. If \( \theta < 0.5 \), then there exists a unique value \( d(\theta, A) \) such that
   - \( c^a > c^b \) for \( d < d(\theta, A) \),
   - \( c^a = c^b \) for \( d = d(\theta, A) \),
   - \( c^a < c^b \) for \( d > d(\theta, A) \).

2. If \( \theta \geq 0.5 \), then \( c^a < c^b \).

**Proof:** From equations (2) and (9) we know that \( k^a < k^b \). At all interior equilibria \( f'(k^a) > 1 \). The function \( f(k) - k \) is increasing for \( k \in (0, f^{-1}(1)) \) and decreasing afterwards.

\[ k = kf'(k) \iff f'(k) = 1. \]
1) Assume $\theta < 0.5$ and define the functions $k^a(d, \theta, A)$ and $k^b(\theta, A)$ to denote the steady state values of capital of the two economies. $k^a(\cdot, \theta, A)$ is a monotonically decreasing function on $(0, \infty)$ into $(0, f^{-1}(1))$. Therefore, for a given $(\theta, A)$, there exists $d(\theta, A) \in (0, \infty)$ such that $f(k^a(d, \theta, A)) - k^a(d, \theta, A) = f(k^b(\theta, A)) - k^b(\theta, A)$. Therefore, $c^a > c^b$ if and only if for $d < d(\theta, A)$. 

2) For $\theta \geq 0.5$, $k^a < k^b < f^{-1}(1)$ and consequently, $c^a < c^b$. \qed

Proposition 8 implies that the asset market leads to lower equilibrium consumption when $\theta \geq 0.5$. This result holds for both equilibria under multiplicity in the economy with the asset market. This rules out a possibility of Pareto improvement through the asset market.

The case when $\theta < 0.5$ is more intriguing. Figure 3 shows equilibrium consumption with and without the asset market. It is shown that the economy with the asset market converges to the golden rule of capital accumulation as $d$ goes to $0$. This implies that for any given $0 < \theta < 0.5$ and $A > 0$, we can find a sufficiently small $d$ so that the equilibrium consumption in the economy with the asset market is higher than without the asset market if $\theta > 0.5$.

Figure 3: Steady state consumption under two regimes: $A = 1, \theta = 0.3, d = 10^{-6}$

Suppose now that the economy without the asset market is in equilibrium and $\theta < 0.5$. Since there is overaccumulation of capital without the asset market when $\theta < 0.5$, the question is whether the economy can achieve a higher welfare by mitigating the overaccumulation...
overaccumulation through the asset market. When the economy in equilibrium opens
the asset market, it converges to the new equilibrium from above since \( k^b > k^a \) and
both \( k^a \) and \( k^b \) are globally stable fixed points. It turns out that the consumption in
transition to the new equilibrium is higher since
\[
\begin{align*}
c_t^a := \theta f(k_t^a) + P(k_t^a) > \theta f(k^a) + P(k^a) &=: c^a > c^b.
\end{align*}
\] (21)
The inequality in equation (21) holds since the functions \( P(k) \) and \( f(k) \) are increasing
in \( k \) and \( k_t^a > k^a \). Figure 4 illustrates the transition from the old to the new equi-
librium. Therefore, when \( \theta < 0.5 \), the economy before opening of the asset market is
dynamically inefficient with overaccumulation of capital. By opening the market it can
achieve a Pareto improvement if \( d \) is sufficiently small. In other words, all generations of
consumers are better off after opening the asset market. Here the assets play the same
role as government debt crowding out investment by moving the economy from a state
of overaccumulation to a state of underaccumulation.

The results obtained so far using a Cobb-Douglas production function generalize to
economies with more general production functions. Suppose we have a more general
production function which satisfies Assumption 1. Let
\[
E_f(k) := \frac{k f'(k)}{f(k)}
\] (22)
denote the elasticity of the production function \( f \). Since \( E_f \) depends on \( k \), the function
\( g(k) \) may not be monotonic. This may imply multiple solutions of \( g(k) = h(k) \). The
value $\varepsilon = E_f(0)$ determines the shape of the time one map $G$ at zero. Figure 5 shows the possibility of multiple REE for a more general production function.

Figure 5: Time one map $G(k)$

Figure 5 shows that the zero REE is stable for $\varepsilon < 0.5$ and unstable for $\varepsilon > 0.5$ as in the Cobb-Douglas case. When $\varepsilon = 0.5$ the zero can be stable and unstable depending on other parameters. Thus, the dynamics of the economy is similar to the Cobb-Douglas case with the exception that multiple REE can arise for any $\varepsilon$ and there may be as many REE as the solutions of the equation $g(k) = h(k)$ plus the zero. It is worth mentioning that the welfare analysis in Section 3.4 holds also for the general case. A more formal analysis of the REE of the general case is provided in appendix.

4 Concluding Remarks

The present paper makes the general point that in an OLG economy opening asset markets too early may result in a poverty trap if the capital share in production is more than one half. This claim seems to contradict the common view on the role of financial markets to channel savings towards productive investment. However, this prediction follows from the premise that paper assets being in fixed supply and unrelated to physical capital. Under these assumptions, the asset market merely absorbs savings which could be otherwise invested in production. This illuminates the aspect that asset investments are inherently unproductive unless the assets are used to fund new investment to facilitate capital accumulation.
Another implication of the results is that an asset market can actually remedy a state of overaccumulation in an economy without asset if the capital share in the production is less than one half. When the proceeds from the exogenous technology are sufficiently small, the economy can achieve a Pareto improvement by opening the asset market. This highlights an important trade off of a policy choice in an economy. It might improve its welfare by opening up the market, but might also be caught in a poverty trap converging to the zero equilibrium. Whether opening up the asset market improves the welfare depends on values of initial capital stocks and other parameters.

A Appendix

A.1 Uniqueness of the Functions $P$ and $G$

We show that there exist functions $P$ and $G$ satisfying equations (12) and (13) and how to approximate them.

Lemma 1 Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ denote a sequence of monotonically increasing functions such that $f_{n+1} \leq f_n$. If $x + f_{n+1}(x) = y + f_n(y)$ then $x \geq y$.

Proof: Suppose $y > x$. Then we have

$$0 < y - x = f_{n+1}(x) - f_n(y) < f_{n+1}(x) - f_n(x) \leq 0,$$

which is a contradiction, i.e., $x \geq y$. □

Lemma 2 Consider the sequence of functions $\{P_n, G_n\}_{n=1}^\infty$ defined in equations (29) and (30). Then
a) $\{P_n, G_n\}_{n=1}^\infty$ is a monotonic sequence of functions, in particular, $0 \leq P_1 \leq P_2 \leq ...$ and $G_1 \geq G_2 \geq ... \geq 0$;
b) $P_n(k)$ and $G_n(k)$ are continuous, and monotonically increasing functions with respect to $k$, for each $n \in \mathbb{N}$;

Proof: We use the induction argument. Firstly we show that claims a) and b) are correct for $n = 1$ and then by assuming that both claims are correct for arbitrary $n$, we
show that they are also correct for $n + 1$. By construction $G_1(k)$ solves the following equation

$$x + S(r(x), 0) = w(k)\quad (24)$$

Using the continuity and the monotonicity property of the price law, it follows from equation (24) that $G_1(k)$ is positive, continuous and monotonically increasing with respect to $k$. From $P_1(k) = S(r(G_1(k)), 0)$, we get that $P_1(k)$ is also positive, continuous and monotonically increasing function.

Now let us assume that claims a) and b) are correct for an arbitrary $n$. By construction $G_n(k)$ and $G_{n+1}(k)$ solve the following equations

$$x + S(r(x), P_{n-1}(x)) = w(k)\quad (25)$$

$$y + S(r(y), P_n(y)) = w(k).\quad (26)$$

Since $P_{n-1} \leq P_n$ and the price law is a monotonic transformation of the price function it follows that for any given $k$, $f_{n-1} \leq f_n$, where $f_{n-1}(x) = S(P_{n-1}(x), r(x))$ and $f_n(y) = S(P_n(y), r(y))$. In addition monotonicity of $P_n(y)$ implies the monotonicity of $f_n(y)$. Now we can apply Lemma 1 in order to get that $G_n \geq G_{n+1}$. The last inequality with the following equations

$$P_n(k) = S(P_{n-1}(r(G_n(k)), G_n(k)))\quad (27)$$

$$P_{n+1}(k) = S(P_n(r(G_{n+1}(k)), G_{n+1}(k)))\quad (28)$$

implies that $P_n \leq P_{n+1}$. This concludes the proof of claim a). By applying the implicit function theorem and using the property of the price law it follows that the solution of equation (26) is a continuous and monotonically increasing function. The continuity and the monotonicity of $P_{n+1}(k)$ follows from equation (28).

□

A.1.1 Existence of REE

**Proposition 8** Consider the sequence of price and capital accumulation functions, \(\{P_n, G_n\}_{n=1}^{\infty}\), defined as follows

a) $P_0 = 0$, 
b) \( k_1 = G_n(k) \) solves the following equation

\[
 k_1 + S(r(k_1), P_{n-1}(k_1)) = w(k),
\]

(29)

c) \( P_n(k) \) is determined by

\[
 P_n(k) = S(P_{n-1}(r(G_n(k)), G_n(k))),
\]

(30)

then the sequence of \( \{P_n, G_n\}_{n=1}^{\infty} \) functions converges uniformly to the equilibrium price and capital accumulation functions, \( \{P, G\} \).

(d) \( \{P, G\} \) are both continuous and monotonically increasing.

**Proof:** Lemma 2 implies a point wise convergence of the sequence, \( \{P_n(k), G_n(k)\}_{n=1}^{\infty} \).

By contraction of this sequence it follows that

\[
 \|P - P_0\| = \sup_{k \in K} |P(k) - P_0(k)| = \sup_{k \in K} |P(k)| < \sup_{k \in K} |w(k)| = M < \infty.
\]

(31)

The price equation with Lemma 2 implies that

\[
 0 < P(k) - P_1(k) = \frac{P(G(k)) + d}{r(G(k))} - \frac{P_0(G_1(k)) + d}{r(G_1(k))} \leq \frac{P(G(k)) - P_0(G(k))}{r(G(k))}.
\]

(32)

Equation (32) implies that

\[
 0 < P(k) - P_n(k) \leq \frac{P(G(k)) - P_{n-1}(G(k))}{r(G(k))} \leq \ldots \leq \frac{P(G^n(k)) - P_0(G^n(k))}{r(G(k))r(G^2(k))\ldots r(G^n(k))}.
\]

(33)

Since we know that in steady state \( r(k) > 1 \), it follows that for sufficiently large \( n \), \( r(G(k))r(G^2(k))\ldots r(G^n(k)) > 1 \) and consequently from (33) it follows that

\[
 \|P - P_n\| = \sup_{k \in K} |P(k) - P_n(k)| \leq \beta_n M,
\]

(34)

where

\[
 \beta_n = \sup_{k \in K} \frac{1}{r(G(k))r(G^2(k))\ldots r(G^n(k))},
\]

(35)

and \( \lim_{n \to \infty} \beta_n = 0 \).

Equations (34) and (35) imply the uniform convergence of \( P_n \). This implies the uniform convergence of \( G_n \) and subsequently the monotonicity and continuity of \( \{P, G\} \).
A.1.2 Proof of Proposition 3

Proof: Suppose there are two different continuous equilibrium maps $P_1, G_1$ and $P_2, G_2$. Then, there exists at least one $k$ such that $P_1(k) \neq P_2(k)$. Let us assume without loss of generality that $P_1(k) > P_2(k)$. This inequality implies that $G_1(k) < G_2(k)$ and consequently the following inequality should be satisfied

$$\frac{P_1(G_1(k)) + d}{r(G_1(k))} > \frac{P_2(G_2(k)) + d}{r(G_2(k))}. \quad (36)$$

Inequality (36) implies that $P_1(G_1(k)) > P_2(G_2(k))$, from which it follows that $G_1^2(k) < G_2^2(k)$. Continuing this process we obtain that for any $n \in \mathbb{N}$,

$$P(G_1^n(k)) > P(G_2^n(k)) \quad \text{and} \quad G_1^n(k) < G_2^n(k). \quad (37)$$

The map $G_i$ for $i = 1, 2$ can have two stable fixed points either 0 or $\hat{k} > 0$. We consider three cases, either a) $\lim_{n \to \infty} G_1^n(k) = \lim_{n \to \infty} G_2^n(k) = 0$, b) $\lim_{n \to \infty} G_1^n(k) = 0$ and $\lim_{n \to \infty} G_2^n(k) = \hat{k}$, or c) $\lim_{n \to \infty} G_1^n(k) = \lim_{n \to \infty} G_2^n(k) = \hat{k}$.

a) If $\lim_{n \to \infty} G_1^n(k) = \lim_{n \to \infty} G_2^n(k) = 0$, then there exists a constant, $N > 1$, such that $G_1^n(k) = (1 - \delta)G_1^{n-1}(k)$, and $G_2^n(k) = (1 - \delta)G_2^{n-1}(k)$ for $n > N$. This implies that the price function is the wage function which contradicts the inequalities in (37).

b) If $\lim_{n \to \infty} G_1^n(k) = 0$ and $\lim_{n \to \infty} G_2^n(k) = \hat{k}_2$, this leads to a contradiction because

$$0 = P_1(0) = \lim_{n \to \infty} P_1(G_1^n(k)) \geq \lim_{n \to \infty} P_2(G_2^n(k)) = P_2(\hat{k}) > 0. \quad (38)$$

c) If $\lim_{n \to \infty} G_1^n(k) = \lim_{n \to \infty} G_2^n(k) = \hat{k}_2$, then for sufficiently large $n$, $P_1(G_1^n(k))$ and $P_2(G_2^n(k))$ can be expressed as

$$P_1(G_1^n(k)) = \frac{d}{r(G_1^n(k))} + \frac{d}{r(G_1^n(k))r(G_1^{n+1}(k))} + \ldots \quad (39)$$

$$P_2(G_2^n(k)) = \frac{d}{r(G_2^n(k))} + \frac{d}{r(G_2^n(k))r(G_2^{n+1}(k))} + \ldots \quad (40)$$

Equations (39) and (40) are in contradiction with the inequalities given in (37).

\[\square\]

A.2 General Production Function Case

To analyze the number of interior REE, the following lemmas describe the behavior of the function $g$. 

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Lemma 3

- If $E_f(0) < 0.5$, then $\lim_{k \to 0} g(k) = \infty$.
- If $E_f(0) = 0.5$, then $\lim_{k \to 0} g(k) = C > 0$.
- If $E_f(0) > 0.5$, then $\lim_{k \to 0} g(k) = 0$.

Proof: If $f(k) = O(k^\varepsilon)$ as $k \to 0$, then $r(k) = O(k^{\varepsilon - 1})$ and $w(k) = O(k^\varepsilon)$. This implies that $g(k) = O(k^{2\varepsilon - 1})$. \qed

Lemma 4

- If $E_f(k) < 0.5$, then $g'(k) < 0$.
- If $E_f(k) = 0.5$, then $g'(k) = 0$.
- If $E_f(k) > 0.5$, then $g'(k) > 0$.

Proof: Since $w'(k) = -kr'(k)$ it follows that
\[
g'(k) = w'(k)r(k) + w(k)r'(k) = -r'(k)kr(k) - w(k)) = -r'(k)f(k)(2E_f(k) - 1). \tag{41}
\]
Assumption 1 with equation (41) implies the claim of the lemma. \qed

Let $N$ be the number of solutions of equation $E_f(k) = 0.5$. Lemma 4 implies that $N$ is the number of the inflection points of the function $g(k)$. Let $n$ denote the number of REE.

Proposition 9

a) If $E_f(0) < 0.5$, then $n$ is even and $2 \leq n \leq N + 2$.

b) If $E_f(0) > 0.5$, then $n$ is odd and $1 \leq n \leq N + 2$.

Proof: An interior REE solves the equation $g(k) = h(k)$.

a) When $E_f(0) < 0.5$, $g(0) > h(0)$ and $g(\bar{k}) < h(\bar{k})$. Since equation $g'(k) = 0$ has $N$ solutions and $h$ is increasing, it follows that equation $g(k) = h(k)$ has at least one but no more than $N + 1$ solutions. In addition, the number of interior REE should be odd. Since there always exists a corner REE where $k = 0$, the claim of the proposition follows.
b) When $E_f(0) > 0.5$, $g(0) < h(0)$ and $g(\bar{k}) < h(\bar{k})$. This means that $g(k) = h(k)$ may not have a solution. Where there exists a solution, the number of solutions is even. Since there always exists a corner REE where $k = 0$, the claim of the proposition follows. □

Proposition 10 Let $E'_f(k) := \frac{k f''(k)}{f'(k)}$. If $E_f(0) = 0.5$ and

- $g(0) > d$, $n$ is even. Furthermore, $2 \leq n \leq N+1$ if $E'_f(0) < -\frac{1}{2}$ and $2 \leq n \leq N+2$ if $E'_f(0) > -\frac{1}{2}$,
- $g(0) \leq d$, $n$ is odd. Furthermore, $1 \leq n \leq N+2$ if $E'_f(0) < -\frac{1}{2}$ and $2 \leq n \leq N+1$ if $E'_f(0) > -\frac{1}{2}$.

Proof: From Lemma 4 we know that $g'(0) = 0$. It is straightforward to calculate that $g''(0) = -r' r (1 + 2E_f'(0))$. Hence,

$$g''(0) > 0 \iff E'_f(0) < -\frac{1}{2},$$

$$g''(0) < 0 \iff E'_f(0) > -\frac{1}{2}. \quad (42)$$

If $E'_f(0) = -\frac{1}{2}$, then $g''(0) = 0$ and we need a higher derivative to determine the behavior of the function $g$ around 0. □

While Propositions 9 and 10 describe the number of REE when $E_f(k) = 0.5$ has finite number of solutions $N$, the following proposition describes the case when $E_f(k) = 0.5$ has infinite number of solutions.

Proposition 11 If $E_f(k) = 0.5$ is the identity on the interval $(0, \bar{k})$ and

- $g(0) > d$, then $n = 2$,
- $g(0) \leq d$, then $n = 1$.

Proof: First we observe that if $E_f(k) = 0.5$ is the identity on the interval $(0, \bar{k})$ then $f(k)$ is a special case of the Cobb-Douglas production function $f(k) := Ak^\varepsilon$ with $\varepsilon = 0.5$ on the same interval. Therefore, this case corresponds to case b) in Proposition 2. □
References


