Supplementary Appendix

Interconnection in network industries

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September 11, 1996

1 Detailed proofs

Proof of Proposition 1: Choose any common retail price $\tilde{p}$ which yields non-negative profits to both firms. We have to find a tariff t such that the firm's choose price p when they price non-cooperatively.

Their non-cooperative prices are determined by the first order conditions

$$\frac{\partial \Pi_1}{\partial p_1} = 0 \qquad (1)$$

$$\frac{\partial \Pi_2}{\partial p_2} = 0 \qquad (2)$$

By A3, there is a unique solution p_1, p_2 to (**??**) and (**??**). Since the firms are identical, $p_1 = p_2 = \tilde{p}$.

The first order conditions (**??**) and (**??**) yield the following system of linear equations

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{\partial X_1}{\partial p_1} + \frac{\partial Y_1}{\partial p_1} t_1 - \frac{\partial Z_1}{\partial p_1} t_2 = 0$$

$$\frac{\partial \Pi_2}{\partial p_2} = \frac{\partial X_2}{\partial p_2} - \frac{\partial Z_2}{\partial p_2} t_1 + \frac{\partial Y_2}{\partial p_2} t_2 = 0$$

or

$$\begin{pmatrix} \frac{\partial Y_1}{\partial p_1} & -\frac{\partial Z_1}{\partial p_1} \\ -\frac{\partial Z_2}{\partial p_1} & \frac{\partial Y_2}{\partial p_1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial X_1}{\partial p_1} \\ -\frac{\partial X_2}{\partial p_2} \end{pmatrix}$$

Since the system is linear in t_1 and t_2, the optimal prices p_1 and p_2 are uniquely determined by the tariffs provided the system is of full rank, that is

$$\Delta = \begin{vmatrix} \frac{\partial Y_1}{\partial p_1} & -\frac{\partial Z_1}{\partial p_1} \\ -\frac{\partial Z_2}{\partial p_2} & \frac{\partial Y_2}{\partial p_2} \end{vmatrix} \neq 0$$

Let $s = s_1$ denote the market share of firm 1. Then $s_2 = 1 - s$ and

$$Y_i = s(1-s)q_j(p_j)$$
$$Z_i = s(1-s)q_i(p_i)$$

Then

$$\frac{\partial Y_i}{\partial p_i} = \frac{\partial s(1-s)}{\partial p_i} q_j(p_j)$$
$$\frac{\partial Z_i}{\partial p_i} = \frac{\partial s(1-s)}{\partial p_i} q_i(p_i) + s(1-s)\frac{\partial q_i}{\partial p_i}$$

and

$$\frac{\partial s(1-s)}{\partial p_i} = (1-2s)\frac{\partial s}{\partial p_i}$$

With identical firms, $s = 1/2$ and therefore

$$\frac{\partial s(1-s)}{\partial p_i} = (1-2s)\frac{\partial s}{\partial p_i} = 0$$

so that

$$\frac{\partial Y_i}{\partial p_i} = 0$$
$$\frac{\partial Z_i}{\partial p_i} = s(1-s)\frac{\partial q_i}{\partial p_i}$$

and

$$\Delta = s^2(1-s)^2 \frac{\partial q_i}{\partial p_i}\frac{\partial q_j}{\partial p_j}$$

Note that the matrix has full rank for some neighbourhood of $s = 1/2$ by continuity. □

Proof of Proposition 2: Define

$$p^* = \arg\max_p \pi(p) = (p-c)q(p)$$

p^* maximizes the total profit over the two networks. It is the price that would be charged if the networks were owned by a single monopolist. endproof

The aggregate profit of two networks is

$$\Pi = \Pi_1 + \Pi_2 = s\pi(p_1) + (1-s)\pi(p_2)$$

In any symmetric equilibrium, $p_1 = p_2 = p$ and the joint profit is

$$\Pi = \Pi_1 + \Pi_2 = s\pi(p) + (1-s)\pi(p) = \pi(p)$$

which is maximized at p^*. By Proposition 1, this symmetric price can be obtained noncooperatively by appropriate choice of symmetric tariff. Using interconnection charges, the duopoly can attain the same result as would be obtained by a monopolist.

Proof of Proposition 2: Let $\Pi_i^* = \Pi_i(p_i^*, p_j^*)$. Then totally differentiating the profit functions with respect to the tariffs gives,

$$\frac{d\Pi_i^*}{dt_i} = \frac{\partial \Pi_i^*}{\partial p_i}\frac{\partial p_i}{\partial t_i} + \frac{\partial \Pi_i^*}{\partial t_i} + \frac{\partial \Pi_i^*}{\partial p_j}\frac{\partial p_j}{\partial t_i}.$$

This decomposes the total effects of a small change in the transfer price into three components. The first is the effect on profits through the firm's own final good price, p_i, holding p_j constant. The second is the immediate impact on the firm's profit, holding both p_i and p_j constant. Third is the effect on profits through the change in the other firm's final good price, p_j, holding p_i constant.

At a Nash equilibrium $d\Pi_i^*/dt_i = 0$ since firm i chooses its transfer price to maximize its profit. Thus,

$$\frac{d\Pi_i^*}{dt_i} = \frac{\partial \Pi_i^*}{\partial p_i}\frac{\partial p_i}{\partial t_i} + \frac{\partial \Pi_i^*}{\partial t_i} + \frac{\partial \Pi_i^*}{\partial p_j}\frac{\partial p_j}{\partial t_i} = 0,\ i, j = 1, 2, i \neq j \tag{3}$$

The Nash equilibrium price locus is characterized by the condition $\partial\Pi_i^*/\partial p_i = 0$. Substituting this into (**??**) and solving the resulting expression for $\partial\Pi_i^*/\partial p_j$, we get

$$\frac{\partial \Pi_i^*}{\partial p_j} = -\frac{\partial \Pi_i^*/\partial t_i}{\partial p_j/\partial t_i} \tag{4}$$

Totally differentiating the profit functions with respect to the other firm's tariff gives (for the same reasons as before)

$$\frac{d\Pi_i^*}{dt_j} = \frac{\partial \Pi_i^*}{\partial p_i}\frac{\partial p_i}{\partial t_j} + \frac{\partial \Pi_i^*}{\partial t_j} + \frac{\partial \Pi_i^*}{\partial p_j}\frac{\partial p_j}{\partial t_j}. \tag{5}$$

Noting again $\partial\Pi_i^*/\partial p_i = 0$ and substituting (**??**) into (**??**) we have

$$\frac{d\Pi_i^*}{dt_j} = \frac{\partial \Pi_i^*}{\partial t_j} - \frac{\partial \Pi_i^*}{\partial t_i}\frac{\partial p_j/\partial t_j}{\partial p_j/\partial t_i}$$

which evaluates to

$$\frac{d\Pi_i^*}{dt_j} = -q_i - q_j\frac{\partial p_j/\partial t_j}{\partial p_j/\partial t_i}. \tag{6}$$

This is clearly negative provided prices are increasing in tariffs.

To show the monotonicity of prices, we totally differentiate the first-order conditions

$$\frac{\partial \Pi_1}{\partial p_1}(p_1, p_2, t_1, t_2) = 0$$
$$\frac{\partial \Pi_2}{\partial p_2}(p_1, p_2, t_1, t_2) = 0$$

with respect to (say) t_1 which yields

$$\frac{\partial^2 \Pi_1}{\partial p_1^2}\frac{\partial p_1}{\partial t_1} + \frac{\partial^2 \Pi_1}{\partial p_1 \partial p_2}\frac{\partial p_2}{\partial t_1} + \frac{\partial^2 \Pi_1}{\partial p_1 \partial t_1} = 0$$
$$\frac{\partial^2 \Pi_2}{\partial p_2 \partial p_1}\frac{\partial p_1}{\partial t_1} + \frac{\partial^2 \Pi_2}{\partial p_2^2}\frac{\partial p_2}{\partial t_1} + \frac{\partial^2 \Pi_2}{\partial p_2 \partial t_1} = 0$$

which can be written as

$$H \begin{pmatrix} \frac{\partial p_1}{\partial t_1} \\ \\ \frac{\partial p_2}{\partial t_1} \end{pmatrix} = - \begin{pmatrix} \frac{\partial^2 \Pi_1}{\partial p_1 \partial t_1} \\ \\ \frac{\partial^2 \Pi_2}{\partial p_2 \partial t_1} \end{pmatrix} \tag{7}$$

where

$$H = \begin{pmatrix} \frac{\partial^2 \Pi_1}{\partial p_1^2} & \frac{\partial^2 \Pi_1}{\partial p_1 \partial p_2} \\ \\ \frac{\partial^2 \Pi_2}{\partial p_2 \partial p_1} & \frac{\partial^2 \Pi_2}{\partial p_2^2} \end{pmatrix}$$

is the Hessian matrix of the profit function.

By assumption A7, H is a *Hicksian* matrix.[1] Furthermore its off-diagonal elements are non-negative, that is

$$\frac{\partial^2 \Pi_1}{\partial p_1 \partial p_2} = \frac{-\partial^2 C_i}{\partial q_i \partial q_j} \frac{\partial q_i}{\partial p_i} \frac{\partial q_j}{\partial p_j} \geq 0$$

by A5 and A6. This implies that H is nonsingular and moreover that the $H^{-1} \leq 0$, that is, all its elements are non-positive (Takayama (1985), Theorem 4.D.3).[2]

Consequently (**??**) can be solved yielding

$$\begin{pmatrix} \frac{\partial p_1}{\partial t_1} \\ \\ \frac{\partial p_2}{\partial t_1} \end{pmatrix} = -H^{-1} \begin{pmatrix} \frac{\partial^2 \Pi_1}{\partial p_1 \partial t_1} \\ \\ \frac{\partial^2 \Pi_2}{\partial p_2 \partial t_1} \end{pmatrix} \tag{8}$$

At the Nash equilibrium $\partial^2 \Pi_1 / \partial p_1 \partial t_1 = 0$ (from equation (**??**)) and $\partial^2 \Pi_2 / \partial p_2 \partial t_1 = -\partial q_2 / \partial p_2 > 0$ (from (**??**)). Recalling that H^{-1} is non-positive, this implies that $\partial p_i / \partial t_1 \geq 0$. Similarly, $\partial p_i / \partial t_2 \geq 0$. This establishes the weak monotonicity of final good prices with respect to transfer prices.

Substituting in (**??**) and recalling A5 we have

$$\frac{d\Pi_i^*}{dt_j} < 0 \qquad i, j = 1, 2, i \neq j \tag{9}$$

Also, the equilibrium tariff rates satisfy the first order conditions

$$\frac{d\Pi_i^*}{dt_i} = 0 \qquad i = 1, 2 \tag{10}$$

Conditions (**??**) and (**??**) show that a simultaneous reduction in tariffs, starting from the Nash equilibrium, increases the profits of both firms. □

Proof of Proposition 3: This proposition is in fact established in the course of proving Proposition 2, in which we showed that both firms could be made better off (compared to the Nash equilibrium) by a simultaneous reduction in tariffs. We also showed (equation (**??**)) that prices were increasing functions of tariffs and hence lower tariffs imply lower prices. □

[1] Its principal minors alternate in sign.

[2] This is a variant of the well-known Frobenius theorem, and is discussed in Takayama (1985) and also Karlin (1959). Specifically, H is known as a *Metzler* matrix.

2 Extension to n firms

For the case of n firms, the profit function of firm i is

$$\begin{aligned}\Pi_i(p_{i2}, p_{i3}, \dots, p_{in}) = &\sum_{j=2}^{n} p_{ij} q_{ij}(p_{ij}) - \sum_{j=2}^{n} C_{ij}(q_{ij}(p_{ij})) \\ &+ \sum_{j=2}^{n} t_{ij} q_{ji}(p_{ji}) - \sum_{j=2}^{n} t_{ji} q_{ij}(p_{ij}) \end{aligned} \tag{11}$$

Proof of Proposition 1: The non-cooperative prices are determined by the first order conditions

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial p_{1j}} &= q_{1j} + p_{1j}\frac{\partial q_{1j}}{\partial p_{1j}} - \frac{\partial C_{1j}}{\partial q_{1j}}\frac{\partial q_{1j}}{\partial p_{1j}} - t_{j1}\frac{\partial q_{1j}}{\partial p_{1j}} = 0 \qquad \forall j \neq 1 \\
\frac{\partial \Pi_2}{\partial p_{2j}} &= q_{2j} + p_{2j}\frac{\partial q_{2j}}{\partial p_{2j}} - \frac{\partial C_{2j}}{\partial q_{2j}}\frac{\partial q_{2j}}{\partial p_{2j}} - t_{j2}\frac{\partial q_{2j}}{\partial p_{2j}} = 0 \qquad \forall j \neq 2 \\
&\vdots \\
\frac{\partial \Pi_n}{\partial p_{nj}} &= q_{nj} + p_{nj}\frac{\partial q_{nj}}{\partial p_{nj}} - \frac{\partial C_{nj}}{\partial q_{nj}}\frac{\partial q_{nj}}{\partial p_{nj}} - t_{jn}\frac{\partial q_{nj}}{\partial p_{nj}} = 0 \qquad \forall j \neq n.
\end{aligned}$$

These can be evaluated at some configuration of prices $\tilde{p}$ and solved for the resulting tariff rates

$$t_{ji} = \frac{q_{ij} + p_{ij}\frac{\partial q_{ij}}{\partial p_{ij}} - \frac{\partial C_{ij}}{\partial q_{ij}}\frac{\partial q_{ij}}{\partial p_{ij}}}{\frac{\partial q_{ij}}{\partial p_{ij}}}, \qquad i \neq j$$

The rest of the proof follows as before. □

Proof of Proposition 2: Totally differentiating the first order condition of firm i with respect to t_{ij} and recalling that $\partial \Pi_i / \partial p_{ij} = 0$ gives

$$\frac{d\Pi_i^*}{dt_{ij}} = q_{ji} + \frac{\partial \Pi_i^*}{\partial p_{ji}}\frac{\partial p_{ji}}{\partial t_{ij}} = 0$$

or

$$\frac{\partial \Pi_i^*}{\partial p_{ji}} = -\frac{q_{ji}}{\partial p_{ji}/\partial t_{ij}}$$

Substituting into

$$\frac{d\Pi_i^*}{dt_{ji}} = -q_{ij} + \frac{\partial \Pi_i^*}{\partial p_{ji}}\frac{\partial p_{ji}}{\partial t_{ji}}$$

yields

$$\frac{d\Pi_i^*}{dt_{ji}} = -q_{ij} - q_{ji}\frac{\partial p_{ji}/\partial t_{ji}}{\partial p_{ji}/\partial t_{ij}}$$

Now recall that

$$\frac{\partial \Pi_i}{\partial p_{ij}} = q_{ij} + p_{ij}\frac{\partial q_{ij}}{\partial p_{ij}} - \frac{\partial C_{ij}}{\partial q_{ij}}\frac{\partial q_{ij}}{\partial p_{ij}} - t_{ji}\frac{\partial q_{ij}}{\partial p_{ij}} = 0$$

so that a change in t_{ji} affects only the determination of p_{ij}. Thus

$$\frac{\partial p_{ji}}{\partial t_{ji}} = \frac{\partial q_{ji}/\partial p_{ji}}{\partial^2 \Pi_j/\partial p_{ij}^2} > 0 \tag{12}$$

and

$$\frac{\partial p_{ji}}{\partial t_{kl}} = 0 \qquad \text{for } k \neq i \text{ or } l \neq j$$

which implies that

$$\frac{d\Pi_i^*}{dt_{ji}} = -q_{ij} < 0$$

Also

$$\frac{d\Pi_i^*}{dt_{ij}} = 0 \qquad \forall j \neq i$$

by assumption of Nash equilibrium and

$$\frac{d\Pi_i^*}{dt_{kl}} = 0 \qquad \text{for } k \neq i, \forall l$$

A simultaneous reduction in tariffs, starting from the Nash equilibrium, increases the profits of all firms. □

Proof of Proposition 3: Follows from Proposition 2 and (**??**). □

Proof of Proposition 4: As before, the regulated price loci are

$$p_{ij}^R = \frac{\partial C_{ij}}{\partial q_{ij}} + t_{ji} \qquad \forall i \neq j.$$

Firms can achieve any prices by appropriate choice of t_{ij}. □

3 references

Karlin, Samuel, *Mathematical Methods and Theory in Games, Programming, and Economics, Vol. I*, (Reading, Mass: Addison-Wesley, 1959).

Takayama, Akira, *Mathematical Economics*, 2nd edn., (Cambridge: Cambridge University Press, 1985).